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THE MATHEMATICS STUDENT

Vol. XVIII MARCH—JUNE 1950 Nos. 1 & 2

BHASKARA'S CALCULATIONS OF THE GNOMON'S SHADOW

BY

D. A. SOMAYAJI, *Bhimavaram.*

The following problems, proposed and solved by Bhaskara in his *Siddhantasiromani*, are characteristic of his work.

(i) To find the gnomon's shadow in a given direction on any day in a place of known latitude.

Bhaskara's formula runs as follows:

पलप्रभा व्यासदळेन निघ्नी दिग्ज्योद्धृता तां पलभां प्रकल्प्य
साध्याक्षजीवाथ तया विनिघ्नी स्वाक्षज्यपाप्तापमशिञ्जिनी च ॥
ताभ्यां दिनार्धद्युतिवद्विदध्याद्भीष्टदिक्स्थे द्युमणौ द्युतिं वा ।

that is, assuming $Rs/\sin a$ to be the equinoctial shadow where s is the original equinoctial shadow, R the radian measured in minutes and equal to 3438 and a the azimuth of the moment, compute the sine of the corresponding latitude, say sin L; then compute sin D from the formula $\sin D = \sin d \sin L/\sin \varphi$ where d is the sun's declination and φ the latitude of the place. Then $L \pm D$ gives us the zenith-distance wherefrom the shadow can be derived.

The equinoctial shadow is that cast by the gnomon on an equinoctial day at noon. Thus $s/12 = \tan \varphi$, the gnomon's length being 12 units. Hence the Hindu sine of the latitude of the place, in terms of the equinoctial shadow would be $\dfrac{Rs}{\sqrt{12^2+s^2}}$ where the Hindu sine is the modern sine multiplied by R.

The formula may be proved as follows according to modern methods.

Let TRQ be the equator, Z the zenith, P the pole and S the Sun's position. ZS, the zenith-distance of the Sun is equal to $ZT \pm ST$ according as the Sun is in the South or North of the Equator. ZT

is the zenith-distance of the Sun when he has the same azimuth while in the Equator and ST the increase in the zenith-distance due

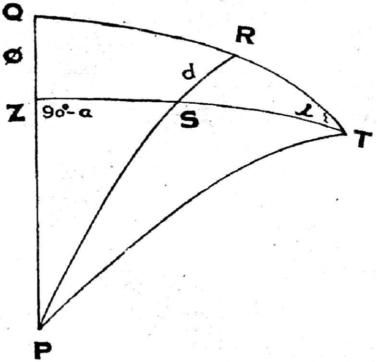

FIG. 1.

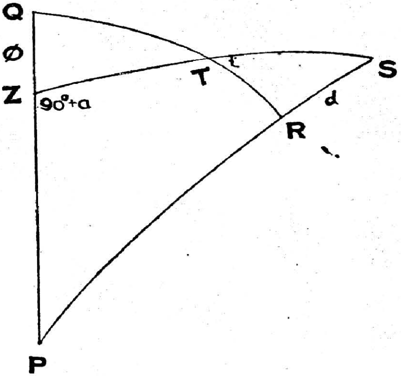

FIG. 2.

to a declination d. In other words, keeping the azimuth constant, ZS is a function of φ and d, and when the declination is nil, the zenith-distance ZT is spoken of as the contribution of φ while ST is looked upon as the contribution of d.

From the triangle PZT

$\cos 90° = \cos ZT \sin \varphi \pm \sin ZT \cos \varphi \sin a,$

where a is the azimuth measured from the East point.

$$\therefore \quad \tan ZT = \pm \frac{\tan \varphi}{\sin a}.$$

But $\tan ZT = S/12$ and $\tan \varphi = s/12$ where S and s are respectively the gnomon's shadows when the Sun is at the points T and Q on the Equinoctial day.

$$\therefore \quad S = \pm \frac{s}{\sin a} \text{ or in the Hindu form } \pm \frac{Rs}{\sin a}.$$

Hence we are asked to multiply the equinoctial shadow by R and divide by the *Dik-jya* or $\sin a$. Taking S to be the equinoctial shadow and finding the sine of the corresponding latitude means finding sin ZT which is equal to

$$\frac{R \times \frac{Rs}{\sin a}}{\sqrt{12^2 + \frac{R^2 s^2}{\sin^2 a}}}$$

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where the denominator is called the *Chāyakarna* or the hypotenuse of the triangle formed by the gnomon and the shadow. Now sin ST may be found as follows.

From triangle ZTQ,

$$\sin \varphi = \sin i \ \sin \mathrm{ZT}$$

and from STR,

$$\sin d = \sin i \ \sin \mathrm{ST}$$

$$\therefore \frac{\sin d}{\sin \varphi} = \frac{\sin \mathrm{ST}}{\sin \mathrm{ZT}} \text{ so that } \sin \mathrm{ST} = \frac{\sin \mathrm{ZT} \times \sin d}{\sin \varphi}.$$

Hence we are asked to multiply sin ZT by *Krantijya* or sin d and divide by *Akshajya* or sin φ. Thus having found sin ZT and sin ST, arcs ZT and TS can be computed and their difference or the sum gives us the zenith-distance of the Sun from which the shadow can be found.

Bhaskara's ingenuity lies in conceiving the proportionality between the sine functions of the arcs ZT, ST and those of φ and d and further in the concept of the individual contributions of ϕ and d.

His proof, however, runs as follows. Let (g) represent the *Trijya-vritta* or the circle with an arbitrary radius R and centre the gnomon's foot g; let gK be the shadow for an azimuth a and gL the

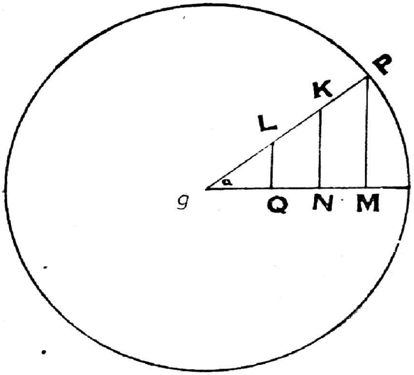

FIG. 3.

shadow in the same direction on an equinoctial day. Then since on an equinoctial day LQ or the *Bhuja* of the shadow as it is called is always equal to the equinoctial shadow s,

$$\therefore \quad g\mathrm{L} = \frac{\mathrm{R}s}{\sin a}$$

where $gP = R$ and PM the Hindu sine of the azimuth. It is obvious that the shadow's extremity traces, on the equinoctial day, a line parallel to the East-West line at a distance of the equinoctial shadow. On other days, the *Bhuja* is obtained as follows. From the triangle PZS, we have

$$\sin d = \cos z \sin \varphi + \sin z \cos \varphi \sin a$$

$$\therefore \quad \frac{12 \sin d}{\cos z \cos \varphi} = 12 \tan \varphi + 12 \tan z \sin a.$$

Here $12 \tan z \sin a$ is the *Bhuja*, $12 \tan \varphi$ the equinoctial shadow and $12/\cos z = K$ the *Chayakarna* so that we have

$$\frac{K \sin d}{\cos \varphi} = s + B.$$

But $\sin d/\cos \varphi$ represents the sine of the rising azimuth of the Sun, for, from the triangle PNS, where P is the pole, N the north point and S the Sun on the horizon, we have $\sin d = \cos \varphi \sin A$ where A is the rising azimuth. Sin A is called *Agrajya* and K sin A or $K \sin d/\cos \varphi$ is termed *Karnâgra* or the *Agrajya* as represented in a circle with the *Chayakarna* as radius.

In other words

$$Karnagra = s + B$$

as given by Bhaskara in the sloka

त्रिभज्याहृताकाग्रका etc. Sl. 72 *Siddhanta Siromani* Ch. 3.

This formula therefore implies in a nutshell the formula of spherical trigonometry

$$\sin d = \cos z \sin \varphi + \sin z \cos \varphi \sin a$$

and throws some light on the *Sankuvedha* or the gnomonics of the Hindus. When a shadow is measured, and its *Bhuja*, the Sun is known and the time of the day as well.

Now reverting to the problem before, we have seen that $gL = Rs/\sin a$ (fig. 3) from which sin ZT can be calculated. gL is said to be '*Ishta-Akshabha* or *Ishta-Drik-mandalapalabha* which means that just as when the declination is zero, we call the shadow of the gnomon when the Sun is on the meridian, the equinoctial shadow pertaining to the vertical of the meridian, similarly on the same equinoctial day, the shadow of the gnomon, when the Sun is on another vertical called *Ishta-Drik-mandala*, is termed Ishta-Drik-mandalapalabha.

The remaining part of the shadow namely LK is due to the Sun's advance in declination. Bhaskara directs us to compute

$$\sin D = \frac{\sin d \sin ZT}{\sin \varphi}$$

evidently bearing in mind figures 1 and 2 wherefrom he must have derived the concept of proportionality mentioned before. Then he directs us to compute the arcs ZT and ST or L and D, so that $L \pm D$ gives us the zenith-distance of the movement wherefrom the shadow gK can be computed.

In another problem, we find Bhaskara using the *calculus of finite differences.* The problem is, however, due to one Sripati, an earlier Astronomer than Bhaskara and the author of *Siddhantasekhara.* The problem is as follows.

To draw the correct East-West line by marking two gnomon's shadows on the same day.

Bhaskara gives the method as follows :

वृत्तेऽम्भस्सुसमीकृतक्षितिगते केन्द्रस्थशङ्कोः क्रमात्
भाग्रं यत्र विशल्यपैति च यतस्तत्रापरैन्द्यौ दिशौ ।
तत्कालापमजीवयोस्तु विवरात् भाकर्णमित्याहतात्
लम्बज्याप्तमिताङ्गुळैरयनदिश्यैन्द्री स्फुटा चालिता ॥

that is, in a circle drawn on a water-levelled or horizontal plane, mark the points where the shadow of the gnomon situated in the centre equals the radius, (in the forenoon and the afternoon). The join of these two points will give us roughly the East-West line. The true East point is removed in the direction of the Sun's movement in declination, by an amount equal to $\frac{K\Delta\,(\sin d)}{\cos \varphi}$ units (where K is the *Chayakarna*, and Δ (sin d) is the variation in sin d between the two moments.)

It may be proved as follows in the language of modern mathematics.

Let g be the gnomon's foot ; FW the rough East-West line. If E be the true East point, it is required to find the variation in the *Bhuja* EA for a small variation in declination. The shadow gE is equal to $\frac{K \sin z}{R}$ in the Hindu form, where K is the *Chayakarna*, so that the *Bhuja* is equal to $\frac{K \sin z \sin a}{R^2}$ where a is the Hindu

azimuth EGA. We have from the spherical Δ PZS as before

$$\sin d = \cos z \sin \varphi + \sin z \cos \varphi \sin a$$

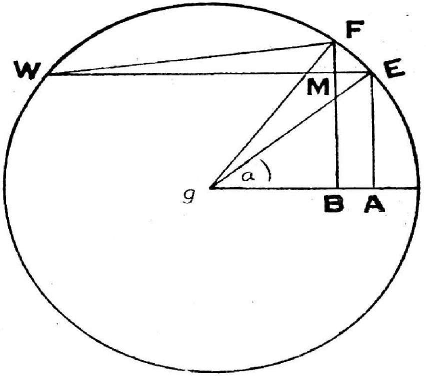

FIG. 4.

or in the Hindu form, $R^2 \sin d = R \cos z \sin \varphi + \sin z \cos \varphi \sin a$ and on differentiation,

$$R^2 \Delta (\sin d) = \sin z \cos \varphi \; \Delta (\sin a),$$

for z has not varied, the lengths of the shadows being the same at the two moments.

$\therefore \quad \sin z \; \Delta (\sin a) = \dfrac{R^2 \; \Delta(\sin d)}{\cos \varphi}$ so that the variation in the *Bhuja* $\dfrac{K \sin z \sin a}{R^2}$ namely

$$\frac{K \sin z \Delta (\sin a)}{R^2} = \frac{K}{R^2} \times \frac{R^2 \; \Delta (\sin d)}{\cos \varphi} = \frac{K \Delta (\sin d)}{\cos \varphi}$$

as given by Bhaskara.

Bhaskara, however, deduces it as follows. He says that the variation in the *Bhuja* is the same as that in the *Karnagra* which differs from the *Bhuja* by the constant equinoctial shadow. But the *Karnagra* is $\dfrac{K \sin d}{\cos \varphi}$ as noticed before, so that the variation therein is $\dfrac{K}{\cos \varphi} \Delta (\sin d)$, for K the *Chayakarna* is constant. Then he proceeds to prove it from first principles. He takes the *Aksha Tribhuja* or latitudinal triangle E′A′B′ where E′,A′,B′ are respectively the projections on the meridian plane of the East point, the Sun's position on the horizon and the Sun's position at the given moment.

It is called an *Aksha Tribhuja* because it is similar to the triangle formed by the gnomon, the equinoctial shadow and the corresponding hypotenuse called the *Akshakarna*, or to the triangle formed by the sine of the latitude or *Akshajya*, the cosine of the latitude called *Lambajya* and the radius of the sphere R.

A note is here necessary about these latitudinal triangles. All spherical triangles formed by the intersection of the Equator or parallels to the Equator with horizon, meridian or prime vertical are

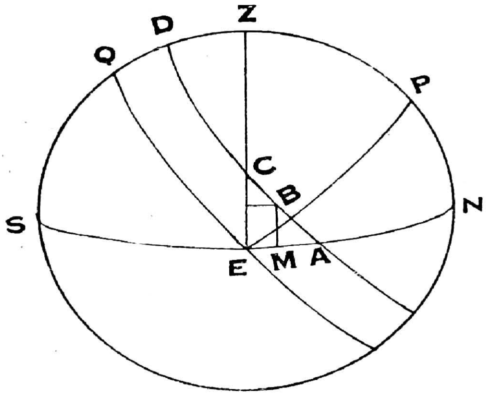

FIG. 5.

termed as such for they arise out of the latitude of the place. In the equatorial places there will be no inclined ascension of the celestial bodies so that no triangles are formed. For ready reference, eight of the important latitudinal triangles are enumerated namely (Ref. fig. 5) (1) The fundamental gnomonic triangle mentioned above, (2) The second latitudinal triangle also mentioned before, and the remaining six being the projections on the meridian plane, of the spherical triangles EAB, EBM, ABM, EBC, AEC, BNC. Even the first two triangles are in the meridian plane, the second being formed by the perpendicular from Q, on the plane of the horizon which is termed *Lambajya* or cos φ, the distance of the foot of the perpendicular from the centre of the sphere equal to sin φ and the radius joining Q to the centre. In practice an armillary sphere is constructed and all these triangles are formed on the plane of the meridian by means of threads and names are given to the various chords. Any problem of the sphere is traced to these fundamental triangles and solved readily.

In the present case, the projection of EAB namely E′A′B′ is taken, E′B′ representing the Hindu sine of the declination, E′A′ the sine of the rising azimuth equal to R sin d /cos φ and A′B′ being called the *Kujya* a sine-chord in the diurnal circle. In view of the similarity of the triangle E′A′B′ to OQR where O is the centre of the sphere and R the foot of the perpendicular from Q on the plane of the horizon, we have the proportion

$$\frac{QR}{OQ}=\frac{E'B'}{E'A'} \text{(E′, of course, coincides with O)}$$

or what is the same $\frac{\cos Q}{R}=\frac{\sin d}{\sin A}$ wherefrom we have

$$\sin A=\frac{R \sin d}{\cos \varphi}$$

obtained according to the Hindu method. What Bhaskara says now in the context is that the principle of similarity can be extended to small variations in the sides of the triangle E′A′B′, which is a modern concept of the calculus of finite differences as applied to the sides of a triangle. To quote his own words

यदि लम्बज्याकोट्या त्रिज्याकर्णः, क्रान्तिज्यान्तरेण किमिति

that is, if by the *Koti* of cos φ, we have the hypotenuse R, what shall we have for the variation in sin d or E′B′? The answer is, he says, the variation in the Agra or E′A′ and is equal to $\frac{R\Delta\ (\sin d)}{\cos \varphi}$. Then he says, if we have that variation in the sphere of radius R, what will it be for the *Chayakarna*? (for the extremity of the shadow lies at the end of *Chayakarna* K and the triangles gHK and KRQ are similar where KQ is the radius of the sphere, gH the gnomon and gK the shadow.) The correction is to be made at the extremity of the shadow., The answer is thus

$$\frac{K}{R}\times\frac{R\Delta\ (\sin d)}{\cos \varphi}=\frac{K\Delta\ (\sin d)}{\cos \varphi}.$$

A FORMULA OF BHASKARA FOR THE CHORD OF A CIRCLE LEADING TO A FORMULA FOR EVALUATING SIN α°

BY

M. G. INAMDAR, *Nagpur*

In the chapter on *Kshetra-Vyavahâra* (topics about the fields) in *Leelâvatée*, (a work on arithmetic of the 12th century A.D.) is given the following verse for finding the chord of a circle when the arc is given :

अथ स्थूल-जीवा-ज्ञानार्थं लघुक्रियया करणसूत्रं (वृत्तम्)

चापोन-निघ्न-परिधिः प्रथमाह्वयः स्यात् ।
पञ्चाहतः परिधि-वर्ग-चतुर्थभागः ॥
आद्योनितेन खलुतेन भजेत् चतुर्घ्न- ।
व्यासाऽऽहतं प्रथम माप्तमिहज्यकास्यात् ॥

Translation : Here is given a practical formula for knowing a chord approximately by a little working (verse).

The conjugate arc (चापोनपरिधिः) multiplied by (निघ्न) the original arc (चाप) should be called प्रथम (आद्य), the first (basic element). One should divide the प्रथम multiplied by four times the diameter, by five times the quarter of the square of the circumference diminished by प्रथम (आद्य). The result so obtained (इह आप्तं) will be the chord, or expressed in symbols

$$\text{chord} = \frac{4 \text{ diameter} \times \text{arc} \times \text{conjugate arc}}{5 \times \dfrac{\text{circumference}^2}{4} - \text{arc} \times \text{conj. arc}}$$

leading to $$\sin\alpha = \frac{4\alpha(\pi-\alpha)}{5\left(\dfrac{\pi}{2}\right)^2 - \alpha(\pi-\alpha)}$$

which is a very easy and elegant formula.

गणेशदैवज्ञ (Astronomer Ganesh of the 16th century) modifies this formula into

$$\sin\alpha = \frac{\left(\dfrac{\pi}{2}\right)^2 - \left(\dfrac{\pi}{2} - \alpha\right)^2}{\left(\dfrac{\pi}{2}\right)^2 + \frac{1}{4}\left(\dfrac{\pi}{2} - \alpha\right)^2}$$

in his commentary **बुद्धिविलासिनी** (*Buddhi Vilāsini*) on this verse.

In order to determine the degree of accuracy of this formula I append a table of values of sin α for different values of α as given by the Bhaskara formula along with their values as accepted to-day.

Value of α	sin α (by Bhaskara formula)	sin α (Modern Value)	Value of α	sin α (by Bhaskara formula)	sin α (Modern Value)
5°	·0883	·0872	45°	·7059	·7071
10°	·1752	·1736	50°	·7647	·7660
15°	·2603	·2588	60°	·8648	·8660
20°	·3431	·3420	70°	·9390	·9397
25°	·4232	·4226	80°	·9846	·9848
36°	·5871	·5878	85°	·9961	·9962

It appears that *at most* the Bhaskara value differs by ·0015 i.e., 1½ in a thousand.

If AB be a chord of a circle and CD the perpendicular diameter cutting AB at E, we see (by writing the area of the triangle ADC in two ways) that

$$\frac{1}{AE} = \frac{\text{diameter}}{AD \cdot AC}.$$

From this, Bhaskara gets the inequality

$$\frac{1}{AE} > \frac{\text{diameter or } `d'}{\text{Arc } AD \times \text{Arc } AC}$$

since the arc is greater than the chord.

Assume

$$\frac{1}{\text{semi-chord}} = P \frac{d}{\text{semi-arc} \times \text{supplementary semi-arc}} + Q. \qquad (1)$$

where P and Q are multipliers (constants) to be determined. They will be determined from the following two particular cases :—

(i) when

$$\text{arc } AD = \frac{c}{12},\ AE = \frac{r}{2} \left[\text{or when arc } ADB = \frac{c}{6},\ \text{chord } AB = r\right]$$

where 'c' means the circumference and r means radius.

$$\therefore \frac{1}{\frac{r}{2}} = P \cdot \frac{2r}{\frac{c}{12} \times \frac{5c}{12}} + Q \text{ or } \frac{2}{r} = P\,\frac{2r \times 144}{5c^2} + Q. \qquad (2)$$

(ii) when arc AD $= \frac{c}{4}$, AE $= r$

$$\therefore \quad \frac{1}{r} = P \cdot \frac{2r}{\frac{c}{4} \times \frac{c}{4}} + Q \quad \text{or} \quad \frac{1}{r} = P \cdot \frac{2r \times 16}{c^2} + Q. \qquad (3)$$

Solving (2) and (3) we get $P = \frac{5c^2}{128r^2}$ and $Q = -\frac{1}{4r}$.

Substituting these values of P and Q in equation (1), we get the Bhaskara result.

The Ganesh formula may be used for the converse problem viz. "Given the sine of an angle, to find the value of the angle itself."

The values of α obtained by this formula differ by 5′ to 8′ from those obtained from modern tables. This is a fair approximation in consideration of the progress of mathematics in the 12th century A.D. and we should really give credit to Bhaskara and Ganesh for the elegant approximations they have given and the easy methods by which they have been obtained.

REMARKS ON BHASKARA'S APPROXIMATION TO THE SINE OF AN ANGLE

BY

A. A. KRISHNASWAMI AYYANGAR

Bhaskara's rational approximation to the sine of an angle quoted in Mr. Inamdar's paper can be written $\sin\frac{\pi}{n} = \frac{16(n-1)}{5n^2-4n+4}$. It is interesting to note that this is *the best rational approximation* that can be devised. Assuming that $\frac{a+bn+cn^2}{a'+b'n+c'n^2}$ reflects the following properties of $\sin\frac{\pi}{n}$, when $n \geqslant 1$ viz.

(i) $\sin\frac{\pi}{n} = \sin\frac{\pi}{m}$ where $m = \frac{n}{n-1}$,

(ii) $\sin\frac{\pi}{n} \rightarrow 0$ as $n \rightarrow \infty$,

(iii) $\sin\frac{\pi}{n} = 0, 1, \frac{1}{2}$ when $n = 1, 2, 6$ respectively,

we have

(iv) $$\frac{(n-1)(\overline{n \cdot a+b-a})}{a'-(b'+2a')n+n^2(a'+b'+c')} \equiv \frac{a+bn+cn^2}{a'+b'n+c'n^2}\text{; and}$$

(v) $c=0, \quad a+b=0, \quad a'+2b'+4c'=a+2b, \quad a'+6b'+36c'=2a+12b.$

Using (v) in (iv) we have $a'+b'=0$, so that the last two equations in (v) may be reduced to $\left.\begin{matrix}4c'-a'=b\\36c'-5a'=10b\end{matrix}\right\}$

and hence $c' = \frac{5}{16}b$, $a' = \frac{b}{4}$ and we get the approximation

$$(n-1) \Big/ \left(\frac{5}{16}n^2 - \frac{1}{4}n + \frac{1}{4}\right) \text{ or } \frac{16(n-1)}{5n^2-4n+4}$$

or more elegantly $\frac{n^2-(n-2)^2}{n^2+\frac{1}{4}(n-2)^2}$,

which is Ganesa's* variant.

Inversely we can express n in terms of $\sin\frac{\pi}{n}$ in the form

$$1 - \frac{2}{n} = \sqrt{\frac{1-\sin\frac{\pi}{n}}{1+\frac{1}{4}\sin\frac{\pi}{n}}}$$

when $n > 2$, a formula which enables us to calculate readily the angle with a given sine.

---

* Ganesh is an Indian Astronomer of the 16th century who wrote much to simplify Astronomy and popularise it.

TO CONSTRUCT AN ALEF-ONE SET

BY

PATTA SESHADRI, *Ajmer*

1. The purpose of this article is to show that *a distinct sequence of positive integers can be found to correspond to each ordinal of the second class.*

2. The method adopted is in essentials the same as that given by Hardy.* But to obtain the sequences corresponding to the limiting numbers, a scheme different from Hardy's is followed. The scheme is given in tabular form in § 3 and the general results derived are given in § 5.

3. The scheme is as follows:

Sequence	Law of Formation	Limit
$\omega+1,\ \omega+2,\ \omega+3,\ \ldots\ldots,\ \omega+r,\ \ldots\ldots$	$T_{r+1}=T_r+1$	$\omega\cdot 2$
$\omega,\ \omega\cdot 2,\ \omega\cdot 3,\ \ldots\ldots,\ \omega\cdot r,\ \ldots\ldots$	$T_{r+1}=T_r+\omega$	ω^2
$\omega,\ \omega^2,\ \omega^3,\ \ldots\ldots,\ \omega^r,\ \ldots\ldots$	$T_{r+1}=T_r\cdot\omega$	ω^ω
$\omega,\ \omega^\omega,\ \omega^{\omega^\omega},\ \ldots\ldots,\ U_r,\ U_{r+1},\ \ldots$ (a)	$U_{r+1}=\omega^{U_r}$	ε_1 (b)
$\varepsilon_1,\ \varepsilon_1^{\varepsilon_1},\ \varepsilon_1^{\varepsilon_1^{\varepsilon_1}},\ \ldots\ldots,\ T_r,\ T_{r+1},\ \ldots$	$T_{r+1}=\varepsilon_1^{T_r}$	ε_2
$\varepsilon_2,\ \varepsilon_2^{\varepsilon_2},\ \varepsilon_2^{\varepsilon_2^{\varepsilon_2}},\ \ldots\ldots,\ T_r,\ T_{r+1},\ \ldots$	$T_{r+1}=\varepsilon_2^{T_r}$	ε_3
$\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \ldots\ldots\ \varepsilon_r,\ \ldots\ldots$	$T_r=\varepsilon_r$	ε_ω
$\varepsilon_\omega,\ \varepsilon_\omega^{\varepsilon_\omega},\ \varepsilon_\omega^{\varepsilon_\omega^{\varepsilon_\omega}},\ \ldots\ldots,\ T_r,\ T_{r+1},\ \ldots$	$T_{r+1}=\varepsilon_\omega^{T_r}$	$\varepsilon_{\omega+1}$
$\varepsilon_{\omega+1},\ \varepsilon_{\omega+1}^{\varepsilon_{\omega+1}},\ \varepsilon_{\omega+1}^{\varepsilon_{\omega+1}^{\varepsilon_{\omega+1}}},\ \ldots,\ T_r,\ T_{r+1},\ \ldots$	$T_{r+1}=\varepsilon_{\omega+1}^{T_r}$	$\varepsilon_{\omega+2}$
$\varepsilon_{\omega+1},\ \varepsilon_{\omega+2},\ \varepsilon_{\omega+3},\ \ldots,\ \varepsilon_{\omega+r},\ \ldots\ldots$	$T_r=\varepsilon_{\omega+r}$	$\varepsilon_{\omega\cdot 2}$

* See Hobson: *Theory of Functions of a Real Variable*, (1907) § 151; pp. 191–4.

(*a*) The *r*th term of this sequence is denoted by U_r

(*b*) Hobson *l.c.* § 60, p. 76.

Sequence	Law of Formation	Limit
$\varepsilon_{\omega}, \varepsilon_{\omega\cdot 2}, \varepsilon_{\omega\cdot 3}, \ldots\ldots, \varepsilon_{\omega\cdot r}, \ldots\ldots$	$T_r = \varepsilon_{\omega\cdot r}$	ε_{ω^2}
$\varepsilon_{\omega}, \varepsilon_{\omega^2}, \varepsilon_{\omega^3}, \ldots\ldots, \varepsilon_{\omega^r}, \ldots\ldots$	$T_r = \varepsilon_{\omega^r}$	$\varepsilon_{\omega^\omega}$
$\varepsilon_{\omega}, \varepsilon_{\omega^\omega}, \varepsilon_{\omega^{\omega^\omega}}, \ldots\ldots, \varepsilon_{U_r}, \ldots\ldots$ (c)	$T_r = \varepsilon_{U_r}$	$\varepsilon_{\varepsilon_1}$
$\varepsilon_{\varepsilon_1}, \varepsilon_{\varepsilon_2}, \varepsilon_{\varepsilon_3}, \ldots\ldots, \varepsilon_{\varepsilon_r}, \ldots\ldots$	$T_r = \varepsilon_{\varepsilon_r}$	$\varepsilon_{\varepsilon_\omega}$
$\omega, \varepsilon_{\omega}, \varepsilon_{\varepsilon_\omega}, \ldots\ldots, T_r, T_{r+1}, \ldots\ldots$	$T_{r+1} = \varepsilon_{T_r}$	$(L\,\varepsilon)_1$

(c) See (a) above.

In general if N_1 is an epsilon-number,

N_2 is to be regarded as the limit of $N_1, N_1^{N_1}, N_1^{N_1^{N_1}}, \ldots\ldots$
N_ω ,, ,, ,, $N_1, N_2, N_3, \ldots\ldots$
(a) $(LN)_1$,, ,, ,, $\omega, N_\omega, N_{N_\omega}, \ldots\ldots$

4. Notation : $\longrightarrow$ stands for 'corresponds to'; $S\{f(n)\}$ means the sequence whose nth term is $f(n)$.

$A(n, p)$ stands for $n^{n^{\cdot^{\cdot^{n}}}}$, the total number of n's occurring in the expression being p;

$A_1(n)$ stands for $A(n, n)$;
$A_2(n)$,, $A\{A_1(n), n\}$;
$A_{r+1}(n)$,, $A\{A_r(n), n\}$.

5. The general results obtained are as follows:

Rule 1. $f(\omega) \longrightarrow S\{f(n)\}$,
where $f(\omega)$ *is any limiting number between* ω *and* ε_1 *and* $f(n)$ *is what* $f(\omega)$ *becomes when* ω *is replaced throughout by* n.

$$\varepsilon_r \longrightarrow S\{A_r(n)\}$$
$$\varepsilon_\omega \longrightarrow S\{A_n(n)\}$$
$$\varepsilon_{\varepsilon_\omega} \longrightarrow S\left\{A_{A_n(n)}(n)\right\}$$
$$(L\,\varepsilon)_1 \longrightarrow S\left\{A_{A_{A_{\cdot_{\cdot_{A_n(n)}}}}\cdots(n)}(n)}(n)\right\},$$

(a) This ordinal is mentioned in § 6.

the total number of A's occurring in this expression being $n-1$. The most general result of this kind is given below:

Rule 2. *If* $\varepsilon_{L_1} \longrightarrow S\{F_1(n)\}$

$\varepsilon_{L_2} \longrightarrow S\{F_2(n)\}$

$\varepsilon_{L_r} \longrightarrow S\{F_r(n)\}$

and $L_1, L_2, \cdots\cdots$ *have limit* L_ω, *then* $\varepsilon_{L_\omega} \longrightarrow S\{F_n(n)\}$ *and*

$$f\{\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots, \varepsilon_{L_r}\} \longrightarrow S[f\{n, F_1(n), F_2(n), \cdots, F_r(n)\}]$$

where $f\{n_1\ F_1(n), \cdots\cdots F_r(n)\}$ *is exactly the same function of* $n, F_1(n), \cdots, F_r(n)$ *that* $f\{\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots, \varepsilon_{L_r}\}$ *is of* $\omega, \varepsilon_{L_1}, \cdots, \varepsilon_{L_r}$.

6. It is now easy to prove the following theorem: *If the method adopted above enables us to find a distinct sequence to correspond to any limiting number between* ω *and* N_1 *(both inclusive), it (the method) will enable us to find a distinct sequence to correspond to any limiting number between* N_1 *and* $(LN)_1$*

Proof. Let $N_1 \longrightarrow S\{A_1\,\phi(n)\}$; it is easy to show that

$N_2 \longrightarrow S\{A_2\,\phi(n)\}$ †

$N_3 \longrightarrow S\{A_3\,\phi(n)\}$

By Rule 2 $\quad N_\omega \longrightarrow S\{A_n\,\phi(n)\}$.

Similarly $\quad N_{N_\omega} \longrightarrow S\left\{A_{A_n\phi(n)}\,\phi(n)\right\}$

and $\quad (LN)_1 \longrightarrow S\left\{A_{A_{\cdot_{\cdot_{\cdot A_n\phi(n)}}}\phi(n)}\,\phi(n)\right\}$

the total number of A's occurring in the expression being $n-1$. Since the functions giving the nth terms are all different, the sequences obtained are all distinct.

7. Applying this and similar theorems and Rule 2 of § 5 we can now show that the method enables us to find a distinct sequence to correspond to any limiting number of the second class. This we shall show by the method of mathematical induction. The theorem states that if we have gone up to N_1, we can certainly go up to $(LN)_1$. But we have already gone up to $(L\varepsilon)_1$. So we can certainly go up to $(LL\varepsilon)_1$ or $(L_2\varepsilon)_1$. Again since we have gone up to $(L_2\varepsilon)_1$, we can go up to $(LL_2\varepsilon)_1$ or $(L_3\varepsilon)_1$, and so on. Now by Rule (2) we can go up to $(L_\omega\varepsilon)_1$ and so on endlessly. The process must give us a sequence to correspond to any limiting ordinal of the second class.

* This number is defined in § 3.

† This means $A\{A_1\phi(n), n\}$. In general $A\{A_r\,\phi(n), n\}$ is denoted by $A_{r+1}\,\phi(n)$.

8. But what about the non-limiting ordinals of the second class? The sequences corresponding to these can be obtained by the application of Rules 1 and 2 of § 5, provided we now take $f(\omega)$ to mean any ordinal between ω and ε_1 and $f(\omega, \varepsilon_{L_1}, \cdots, \varepsilon_{L_r})$ to mean any ordinal between ε_{L_1} and ε_{L_r}.

9. We can now prove another important theorem: *If r_1 and r_2 are any two ordinals of the second class, the sequences corresponding to them are different.* We have to consider the following cases:

(i) r_1 and r_2 are functions of exactly the same set of limiting numbers;

(ii) r_1 and r_2 are functions of two sets having no common member.

(iii) r_1 and r_2 are functions of two sets of limiting numbers having only some members in common.

Case (i) Let $r_1 = f_1(\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots, \varepsilon_{L_r})$

and $r_2 = f_2(\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots, \varepsilon_{L_r})$

where f_1 and f_2 are different functions.

Now by Rule 2, $r_1 \longrightarrow S\,[f_1\{n, F_1(n), F_2(n), \cdots\cdots, F_r(n)\}]$

and $r_2 \longrightarrow S\,[f_2\{n, F_1(n), F_2(n), \cdots\cdots, F_r(n)\}]$

Clearly the nth terms of the sequences are different functions of n hence the sequences are distinct.

Case (ii) Let $r_1 = f\,\{\varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots, \varepsilon_{L_r}\}$

and $r_2 = f'\{\varepsilon_{L'_1}, \varepsilon_{L'_2}, \cdots\cdots, \varepsilon_{L'_s}\}$

where $\varepsilon_{L'_s} \longrightarrow S\,\{F_s'(n)\}$.

Then $r_1 \longrightarrow S\,[f\,\{F_1(n), F_2(n), \cdots\cdots, F_r(n)\}]$

and $r_2 \longrightarrow S\,[f'\{F_1'(n), F_2'(n), \cdots\cdots, F_s'(n)\}]$.

We see again that the two sequences are distinct.

Case (iii) Let $r_1 = f\,\{\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots \varepsilon_{L_r}, \varepsilon_{L''_1}, \varepsilon_{L''_2}, \cdots\cdots, \varepsilon_{L''_t}\}$;

where $\varepsilon_{L''_t} \longrightarrow S\,\{F''_t(n)\}$

and $r_2 = f'\,\{\omega, \varepsilon_{L_1}, \varepsilon_{L_2}, \cdots\cdots, \varepsilon_{L_r}, \varepsilon_{L'_1}, \varepsilon_{L'_2}, \cdots\cdots, \varepsilon_{L'_s}\}$

Then $r_1 \longrightarrow S\,[f\,\{n, F_1(n), F_2(n), \cdots\cdots, F_r(n), F_1''(n), \cdots\cdots, F_t''(n)\}]$

and $r_2 \longrightarrow S\,[f'\{n, F_1(n), F_2(n), \cdots\cdots, F_r(n), F_1'(n), \cdots\cdots, F_s'(n)\}]$

Once more we see that the two sequences are different.

CONGRUENCE PROPERTIES OF $\sigma(n)$

BY

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1. It was shown by K. G. Ramanathan (1) that $\sigma(n)$, which is defined as the sum of the divisors of an integer n, satisfies the congruence properties

$$\sigma(jn-1) \equiv 0 \pmod{j}$$

where $j = 3, 4, 6, 8, 12$ or 24; and it was shown by Hansraj Gupta (2) that these are the only congruences of this type. Ramanathan's proof makes us of elliptic theta functions and allied infinite series. I give here very elementary proofs of these congruence properties.

2. The congruences can be written explicitly thus:

$$\sigma(3n+2) \equiv 0 \pmod{3}; \qquad \sigma(4n+3) \equiv 0 \pmod{4}$$
$$\sigma(6n+5) \equiv 0 \pmod{6}; \qquad \sigma(8n+7) \equiv 0 \pmod{8}$$
$$\sigma(12n+11) \equiv 0 \pmod{12}; \qquad \sigma(24n+23) \equiv 0 \pmod{24}.$$

To prove the first, we need only observe that if d and d' are conjugate diameters of $3n+2$, i.e. $dd' = 3n+2$, and if one of them is of the form $3k+1$, the other must be of the form $3k'+2$, for then only can dd' be of the form $3n+2$. Also no divisor of $3n+2$ is of the form $3k$. Now $d+d' = 3k+1+3k'+2 \equiv 0 \pmod{3}$, so that the sum of every pair of conjugate divisors of $3n+2$ is $\equiv 0 \pmod{3}$.

Hence $$\sigma(3n+2) \equiv 0 \pmod{3}.$$

Exactly similar proofs hold for the other congruences. I shall only give a proof for the last one: viz. $\sigma(24n+23) \equiv 0 \pmod{24}$. Let $dd' = 24n+23$. Then d is of one of the following possible forms:

$$24k+1,\ 24k+5,\ 24k+7;\ 24k+11;$$
$$24k+13;\ 24k+17;\ 24k+19;\ 24k+23.$$

If d is of the form $24k+1$, then d' must be of the form $24k'+23$, for $dd' = 24n+23$.

Similarly if
$$d = 24k+5, \qquad d' = 24k'+19;$$
$$d = 24k+7; \qquad d' = 24k'+17$$
$$d = 24k+11; \qquad d' = 24k'+13.$$

In each case $d+d' \equiv 0 \pmod{24}$, and hence the result.

3. The result that these are the only congruences of this type, due to Gupta (2), can also be proved easily. Gupta's result can be stated thus:

Theorem $\sigma(jn-1) \equiv 0 \pmod{j}$, $j \geqslant 3$, *for every* $n > 0$ *if and only if* $a^2 \equiv 1$ (mod. j) *for every* a *less than and prime to* j. (A)

The condition (A) implies that the sum of every pair of conjugate divisors of $N = jn - 1$ is $\equiv 0 \pmod{j}$. For let $d = jm + a$ be a divisor of N, $(a < j)$. It is clear that a must be prime to j. Now the conjugate divisor d' is of the form $d' = jm' - a$, for

$$(jm + a)(jm' - a) \equiv 0 \pmod{j}$$

since by assumption, $a^2 \equiv 1 \pmod{j}$.

Thus the condition (A) is sufficient. The necessity of the condition is easily proved, as Gupta has done, by showing that if condition (A) fails, we can find a number of the form $jn - 1$ for which the congruence $\sigma(jn-1) \equiv 0 \pmod{j}$ is false. It can also be easily shown that 3, 4, 6, 8, 12, 24 are the only values of $j \geqslant 3$ for which (A) holds.

References

1. K. G. Ramanathan: Congruence Properties of $\sigma(n)$. p. 33 *Math. Student* Vol. XI No. 1.

2. Hansraj Gupta: do. do. Vol. XIII No. 1.

AN ORDER RESULT RELATING TO THE NUMBER OF DIVISORS OF n IN AN ARITHMETIC PROGRESSION

BY

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1. The object of this paper is to establish the THEOREM: *If $d'(n)$ denotes the number of divisors of n, in the form $ar+b$ $(r=0, 1, 2, \cdots)$, then*

$$\sum_{n=1}^{x} d'(n) = \frac{x \log x}{a} + \alpha x + O(\sqrt{x}), \textit{ where } \alpha \textit{ is a constant } \dagger,$$

and to deduce from it a classical result in the Theory of Numbers, namely,

$$\sum_{n=1}^{x} d(n) = x \log x + (2\gamma - 1)\, x + O(\sqrt{x}),$$ where $d(n)$ is the total number of divisors of n, and γ is Euler's constant.‡

For the proof, we require the known results stated below.

Lemma:—If $a>0$, $b>0$, then

$$\sum_{(r=0)}^{(n-1)} \frac{1}{b+ra} = \frac{1}{a}\log \left\{ b+(n-1)\,a \right\} + l + O\left(\frac{1}{n}\right)$$

where l is a constant. (1·1)

Lemma:—$\sum_{r=1}^{n} \frac{1}{r} = \log n + \gamma + O\left(\frac{1}{n}\right)$, γ being Euler's constant. (1·2)

Proof of the theorem.

Clearly $$\sum_{n=1}^{x} d'(n) = \left[\frac{x}{b}\right] + \left[\frac{x}{b+a}\right] + \cdots + \left[\frac{x}{b+ra}\right] + \cdots + \left[\frac{x}{b+\lambda a}\right],$$

where $\lambda =$ the highest value of r for which $b+ra \ngtr x$, and $[x] =$ the integral part of x.

Now $$\left[\frac{x}{b+ra}\right] \geqq 1, \text{ and } 2 > \frac{x}{b+ra} \geqslant 1 \text{ if } \frac{1}{a}\left(\frac{x}{2} - b\right) < r \leqslant \frac{1}{a}(x-b).$$

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* I am indebted to Dr. Hansraj Gupta for his help in connection with this paper.

† This result was proposed as a problem (No. 1541) for solution by Dr. S. S. Pillai in *Jour. Ind. Math. Soc.* (1929—30), p. 48 where $\frac{x \log x}{b}$ was printed for $\frac{x \log x}{a}$ by mistake. I am not aware of a published solution.

‡ See G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers* 2nd Ed. (1945), p. 262.

Hence $\left[\frac{x}{b+ra}\right]=1$ if $\left[\frac{1}{a}\left(\frac{x}{2}-b\right)\right]+1\leqslant r\leqslant\left[\frac{1}{a}(x-b)\right]$.

Similarly $\left[\frac{x}{b+ra}\right]=2$ if $\left[\frac{1}{a}\left(\frac{x}{3}-b\right)\right]+1\leqslant r\leqslant\left[\frac{1}{a}\left(\frac{x}{2}-b\right)\right]$, and so on.

Thus $\left[\frac{x}{b+ra}\right]=(m-1)$ if $\left[\frac{1}{a}\left(\frac{x}{m}-b\right)\right]+1\leqslant r\leqslant\left[\frac{1}{a}\left(\frac{x}{m-1}-b\right)\right]$

and $\left[\frac{x}{b+ra}\right]\geqslant m$ for all values of $r\leqslant\left[\frac{1}{a}\left(\frac{x}{m}-b\right)\right]$.

Let $m=[\sqrt{x}]$, $k=\left[\frac{1}{a}\left(\frac{x}{m}-b\right)\right]$. Then $m=\sqrt{x}-f_1$ and

$$k=\frac{1}{a}\left(\frac{x}{m}-b\right)-f_2 \text{ where } 0\leqslant f_1,\ f_2<1. \tag{1·3}$$

Also let $S=\sum_{r=0}^{k}\left[\frac{x}{b+ra}\right]$. (1·4)

It is easy to see that $m=O(\sqrt{x})$ and $k=O(\sqrt{x})$. (1·5)

and
$$S=\sum_{r=0}^{k}\frac{x}{b+ra}+O(k+1)$$
$$=x\left\{\frac{1}{a}\log(b+ka)+l+O\left(\frac{1}{k+1}\right)\right\}+O(k+1) \text{ by (1·1)}$$
$$=x\left\{\frac{1}{a}\log\left(\frac{x}{m}-af_2\right)+l+O\left(\frac{1}{k+1}\right)\right\}+O(k+1) \text{ from (1·3)}$$
$$=\frac{x}{a}\log\left(\frac{x}{m}-af_2\right)+lx+O(\sqrt{x}) \text{ since } k \text{ is of order } \sqrt{x}. \tag{1·6}$$

Now

$$d'(n)=S+(m-1)\left\{\left[\frac{1}{a}\left(\frac{x}{m-1}-b\right)\right]-\left[\frac{1}{a}\left(\frac{x}{m}-b\right)\right]\right\}$$
$$+(m-2)\left\{\left[\frac{1}{a}\left(\frac{x}{m-2}-b\right)\right]-\left[\frac{1}{a}\left(\frac{x}{m-1}-b\right)\right]\right\}$$
$$+\cdots+2\left\{\left[\frac{1}{a}\left(\frac{x}{2}-b\right)\right]-\left[\frac{1}{a}\left(\frac{x}{3}-b\right)\right]\right\}$$
$$+1\left\{\left[\frac{1}{a}(x-b)\right]-\left[\frac{1}{a}\left(\frac{x}{2}-b\right)\right]\right\}$$
$$=S-(m-1)\left[\frac{1}{a}\left(\frac{x}{m}-b\right)\right]+\left[\frac{1}{a}\left(\frac{x}{m-1}-b\right)\right]$$
$$+\cdots+\left[\frac{1}{a}\left(\frac{x}{2}-b\right)\right]+\left[\frac{1}{a}(x-b)\right]$$

$$= S - mk + \sum_{s=1}^{m} \left[\frac{1}{a}\left(\frac{x}{s} - b\right)\right]$$

$$= S + \sum_{s=1}^{m} \frac{1}{a}\left(\frac{x}{s} - b\right) + O(m) - mk$$

$$= S + \frac{x}{a}\sum_{s=1}^{m}\frac{1}{s} - \frac{mb}{a} - mk + O(m)$$

$$= S + \frac{x}{a}\left\{\log m + \gamma + O\left(\frac{1}{m}\right)\right\} - \frac{m(b+ka)}{a} + O(m) \text{ by } (1\cdot2)$$

$$= S + \frac{x}{a}\log m + \gamma\frac{x}{a} + O(\sqrt{x}) - \frac{x}{a} + mf_2 \text{ by } (1\cdot3) \text{ and } (1\cdot5)$$

$$= \frac{x}{a}\log(x - af_2m) + \left(l + \frac{\gamma - 1}{a}\right)x + O(\sqrt{x}) \text{ by } (1\cdot5) \text{ and } (1\cdot6)$$

$$= \frac{x}{a}\log x + \frac{x}{a}\log\left(1 - \frac{af_2m}{x}\right) + \alpha x + O(\sqrt{x})$$

where $\alpha = l + \dfrac{\gamma - 1}{a}$ (1-7)

$$= \frac{x}{a}\log x + \frac{x}{a}\left\{-\frac{af_2\,m}{x} - \left(\frac{af_2\,m}{x}\right)^2 - \cdots\right\} + \alpha x + O(\sqrt{x})$$

$$= \frac{x}{a}\log x + \alpha x + O(\sqrt{x}).$$

Corollary. If $a = b = 1$, then $l = \gamma$, $\alpha = 2\gamma - 1$ from (1·7), and $d'(n) = d(n)$.

Hence $\sum_{n=1}^{x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x})$.

THE SPHERICAL BERTRAND CURVE

BY

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1. A curve whose principal normals are also the principal normals of another curve is called a Bertrand curve, and each of these curves is the conjugate of the other. The curvature $\frac{1}{\rho}$ and torsion $\frac{1}{\sigma}$ of a Bertrand curve are connected by the linear relation

$$\frac{A}{\rho} + \frac{B}{\sigma} = 1. \tag{1}$$

For a spherical curve, i.e., for a curve on a sphere, we have

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left[\sigma \frac{d\rho}{ds}\right] = 0. \tag{2}$$

If $\rho, \sigma, s,$ refer to a given Bertrand curve and ρ_1, σ_1, s_1 to the conjugate Bertrand curve at the corresponding point, we have the relations [1]

$$\sigma\sigma_1 = l^2 \operatorname{cosec}^2 \varphi = k, \text{ say} \tag{3}$$

$$\left(1 - \frac{l}{\rho}\right)\left(1 + \frac{l}{\rho_1}\right) = \cos^2 \varphi \tag{4}$$

$$\frac{ds_1}{ds} = \frac{\sigma_1}{l \operatorname{cosec} \varphi} \tag{5}$$

where l is the constant distance between corresponding points of the two curves and φ, the angle between tangents at corresponding points.

Equation (2) combined with (3) and (5) gives

$$\frac{\rho}{\sigma} + \frac{d}{ds}\left[\frac{k}{\sigma_1} \cdot \frac{d\rho}{ds_1} \cdot \frac{\sigma_1}{l \operatorname{cosec} \varphi}\right] = 0$$

i.e. $$\frac{\rho}{\sigma} + \frac{d}{ds_1}\left(\frac{d\rho}{ds_1}\right) \frac{\sigma_1}{l \operatorname{cosec} \varphi} \cdot \frac{k}{l \operatorname{cosec} \varphi} = 0$$

i.e. $$\frac{\rho}{\sigma} + \frac{d^2\rho}{ds_1^2} \cdot \sigma_1 = 0$$

$$\therefore \quad \frac{d^2\rho}{ds_1^2} + \frac{\rho}{k} = 0$$

Hence $\rho = a \sin\left(\frac{s_1}{\sqrt{k}} + \alpha\right)$ where a and α are any constants. We thus have the elegant result;

The Spherical Bertrand Curve is given by

$$\left.\begin{aligned} \rho &= a \sin\left(\frac{t}{\sqrt{k}} + \alpha\right) \\ \frac{A}{\rho} + \frac{B}{\sigma} &= 1 \end{aligned}\right\} \tag{6}$$

where t is a parameter having the interpretation that it is the arc of the conjugate curve.

Thus ρ, σ as well as ρ_1 and σ_1 which are determined by equations (3), (4) and (6), are periodic functions of t.

2. The relation between ρ and s where s is the arc of the given curve, though complicated can be expressed in finite form.

Equations (1) and (2) easily give on reduction

$$2\rho'' - \frac{2A\rho'^2}{\rho(\rho - A)} + \frac{2(\rho - A)^2}{B^2\rho} = 0.$$

dashes denoting differentiation w.r.t. s. Putting $y = \rho'^2$, on integration we get

$$y^{-\frac{1}{2}} = \frac{ds}{d\rho} = \frac{B\rho}{\rho - A} \cdot \frac{1}{\sqrt{c^2 - \rho^2}} \tag{7}$$

where c is some constant of the form Bc', $(c' > 0)$.

The integration of (7) depends upon the relative values of A and c. We have

$$\frac{s}{B} + s_0 = \sin^{-1}\frac{\rho}{c} + \frac{A}{(A^2 - c^2)^{\frac{1}{2}}} \cdot \sin^{-1}\left\{\frac{c^2 - A\rho}{c(\rho - A)}\right\} \tag{8a}$$

when $A^2 > c^2$

$$\frac{s}{B} + s_0 = \sin^{-1}\frac{\rho}{A} - \left(\frac{A + \rho}{A - \rho}\right)^{\frac{1}{2}} \tag{8b}$$

when. $A^2 = c^2$,

$$\frac{s}{B} + s_0 = \sin^{-1}\frac{\rho}{c} - \frac{A}{(c^2 - A^2)^{\frac{1}{2}}} \cosh^{-1}\left\{\frac{c^2 - A\rho}{c(\rho - A)}\right\} \tag{8c}$$

when $A^2 < c^2$.

The relation between s and σ can be written down by using (1) and (8).

3. It may be observed that when a given Bertrand curve lies on a sphere, its conjugate curve cannot be spherical except in the trivial case when the conjugate coincides with the given curve.

Equations (6) and (3) give σ_1 in terms of s_1.

$$\sigma_1 = \frac{k}{B}\left[1 - \frac{A}{a} \operatorname{cosec}\left(\frac{s_1}{\sqrt{k}} + \alpha\right)\right]. \tag{9}$$

The relation between ρ_1 and σ_1, corresponding to the relation (1) between ρ and σ, is

$$-\frac{A}{\rho_1}+\frac{B}{\sigma_1}=1. \tag{10}$$

The condition that this curve should be spherical is

$$\frac{\rho_1}{\sigma_1}+\frac{d}{ds_1}\left(\sigma_1\frac{d\rho_1}{ds_1}\right)=0.$$

which on eliminating ρ_1 with the help of (10) becomes

$$\frac{1}{\sigma_1-B}-\frac{d}{ds_1}\left\{\frac{B\sigma_1\frac{d\sigma_1}{ds_1}}{(\sigma_1-B)^2}\right\}=0 \tag{11}$$

By differentiating (9) w.r.t. s_1, it will be found that σ_1 cannot satisfy the condition (11).

4. If $A=0$ in (1), we have $\sigma=B$ and it is well known that in this case the conjugate curve coincides with the given curve.* Considered as a Bertrand curve, therefore, the curve of constant torsion has no significance. But the discussion of §§ 1 and 2 lead us to the spherical curve of constant torsion.

Putting $A=0$ in (8*c*), or putting $t=s$ in (6), we obtain the following :

The spherical curve of constant torsion B is given by

$$\rho=a\sin\frac{s}{B}\cdot \tag{12}$$

The result confirms and supplements the remark made by Forsyth† that "no algebraic curve of non-zero torsion exists on a sphere." It would be interesting to discuss the shape of the curves given by (6) and (12). ρ and ρ_1 are periodic functions of s_1 for the spherical Bertrand curve, and ρ is a periodic function of s for a spherical curve of constant torsion, taking positive as well as negative values and having definite maxima and minima. In the latter case when $s=2\pi nB$ where n is an integer, we may expect the shape of a spiral converging to a point, and, since $(2\pi nB\pm\varepsilon)$ give opposite signs for ρ, this spiral must be followed by another spiral round the same point in the opposite direction.

References

* Weatherburn :—*Differential Geometry* §§ 12, 30 & 32.

† Forsyth : *Differential Geometry* page 27.

A bibliography of papers dealing with curves of constant torsion will be found in Darboux : *Theorie des Surfaces*.

ORTHOPOLAR THEORY & FEUERBACH'S THEOREM

BY

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The object of this paper is to introduce and prove some of the extensions of Feuerbach's Theorem by applying the Orthopolar theory of Mr. V. Ramaswami Ayyar.[1] Let us state some theorems.

(1) *The orthopole of a st. line w.r.t. a triangle △ is the point at which the pedal lines of the extremities of that line w.r.t. △ meet.*

(2) *The pedal circles of all points on a st. line w.r.t. △ are orthogonal to the orthopolar circle.*[2]

Applying the abovementioned theorems we can prove some interesting lemmas.

(3) *The pedal circles of all points on a diameter w.r.t. △. are concurrent at a point on the N.P. ⊙le.*

The triangle is △ ABC. l is a diameter. Let x, y, z be the feet of ⊥r s on l from A, B and C respectively. Let A′, B′, C′ be the midpts. of BC, AC, AB respectively. Let O be the orthopole of l w.r.t. △ ABC.

[3] O is the image of x thro' B′C′

" y " A′C′

" z " A′B′

∴ C′Y = C′X = C′O.

The orthopolar circle of l w.r.t. △ ABC becomes a pt. circle. So it lies upon the pedal circle of any pt. on l. In particular the N. P. circle also passes thro' O.

(4) *If two isogonal conjugates of △ are collinear with the circumcentre, their common pedal circle touches the N. P. circle.*

Let P and Q be two isogonal conjugates. PQ is a diameter. We know that P and Q have a common pedal circle. If we consider P and Q as points that are noncollinear with S, we show that their pedal circle has two meeting pts. with the N. P. circle, namely the orthopoles of PS and QS. Now when PQS is a st. line the orthopoles coincide and the circles have only one common pt. Thus M'cay's Theorem holds good.

(5) *Feuerbach's Theorem. The N. P. circle touches the in–and ex-circles of the triangle.*

Let the triangle be ABC, with circumcentre S and in-centre I and ex-centre opposite BC, as I_1.

In the case of I, the isogonal conjugates coincide. The incircle becomes the pedal circle of I. In the light of (4), the incircle touches the N. P. circle at the orthopole of IS w.r.t. △ ABC. Thus the latter part of the theorem also holds good. The external Feuerbach pt. is the orthopole of I_1 S.

(6) *If SI is ∥ to a side BC, the internal Feuerbach pt. is the midpt. of the line joining A and the orthocentre.*

Let Y be the foot of ⊥r on SI from B. Let F be the image of Y through A′C′. Let O be the orthocentre. Let A′ be the midpt. of BC. F is the Feuerbach pt.

BYC′SA′ is a cyclic figure.[2]

YC′ ⊥r C′A′;

∴ A′F is the diameter of the N.P. circle.

∴ F is the midpt. of OA.

(7) *If the line SI is antiparallel to BC, the Feuerbach pt. is on the perpendicular bisector of BC.*

AS ⊥r SI ∴ the orthopole F of SI lies upon the ⊥r to BC from S.[3] Hence our statement holds good.

(8) *Corollary. If a line is a diameter of the circumcircle the pedal lines of the extremities w.r.t. △ meet at right angles upon the pedal circle of any pt. on the diameter.* In particular they meet on the N. P. circle.

I am indebted to Dr. M. Venkataraman for valuable suggestions.

References

[1] V. Ramaswami Ayyar *J.I.M.S.*, Vols. 3 and 4. Orthopolar theory.
[2] K. Rangaswami, *Math. Student*, Vol. 6.
[3] Vepa Ramesam and V. P. Sarathy " A Supplement to Ely. Geometry."

A SIMPLE METHOD OF SOLVING D'ALEMBERT'S EQUATION $\Box^2 \phi = -\lambda$

BY

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The solution of D'Alembert's equation is of great importance in the mathematical formulation of Huygen's principle and electro-dynamics. In fact we have for the two fundamental potentials A (the vector potential) and ϕ (the scalar potential), the following equations :

$$\left.\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -4\pi\rho \\ \nabla^2 \mathrm{A} - \frac{1}{c^2}\frac{\partial^2 \mathrm{A}}{\partial t^2} &= -4\pi\frac{j}{c} \end{aligned}\right\} \tag{1}$$

where the symbols ρ, j, and c have the usual meaning.

(1) would be conveniently written as

$$\Box^2 f = -\lambda. \tag{2}$$

Several sophisticated methods of solution of (2) have been offered. One can obtain the solution more generally as

$$f(r, t) = \int \mathrm{G}(r', t'; r, t)\,\lambda(r', t')\,d\tau'\,dt'.$$

where G (r, t), the Green's function satisfies the differential equation

$$\Box^2 \mathrm{G}(r, t) = -\delta(r)\,\delta(t),$$

$\delta(r)$ and $\delta(t)$ being the Dirac delta functions in space and time.

We now propose to give a simple solution to the D'Alembert's equation.

To solve

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\lambda,$$

introduce the transformation

$$\left.\begin{aligned} &x' = x,\ y' = y,\ z' = z,\ \mathrm{R} = r \\ &\mathrm{T} = \left(t \mp \frac{r}{c}\right) \\ &r^2 = x^2 + y^2 + z^2 \end{aligned}\right\} . \tag{3}$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \mp \frac{1}{c}\left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial \mathrm{T}}$$

$$= \left(\frac{\partial}{\partial x'} \mp \frac{x}{cr}\,\frac{\partial}{\partial \mathrm{T}}\right) = \left(\frac{\partial}{\partial x'} \mp \frac{x'}{c\mathrm{R}}\,\frac{\partial}{\partial \mathrm{T}}\right),$$

similarly for $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, etc.

We therefore have

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \left(\frac{\partial}{\partial x'} \mp \frac{x'}{c\mathrm{R}}\,\frac{\partial}{\partial \mathrm{T}}\right) \\ \frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial y'} \mp \frac{y'}{c\mathrm{R}}\,\frac{\partial}{\partial \mathrm{T}}\right) \\ \frac{\partial}{\partial z} &= \left(\frac{\partial}{\partial z'} \mp \frac{z'}{c\mathrm{R}}\,\frac{\partial}{\partial \mathrm{T}}\right) \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \mathrm{T}} \end{aligned}\right\} . \tag{4}$$

Also

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial x'^2} + \left(\frac{x'}{c\mathrm{R}}\right)^2 \frac{\partial^2\phi}{\partial \mathrm{T}^2} \mp \frac{1}{c}\left\{\frac{\partial}{\partial x'}\left(\frac{x'}{\mathrm{R}}\,\frac{\partial\phi}{\partial \mathrm{T}}\right) + \frac{x'}{\mathrm{R}}\,\frac{\partial^2\phi}{\partial \mathrm{T}\partial x'}\right\} \tag{5}$$

with similar expressions for $\frac{\partial^2\phi}{\partial y^2}$, $\frac{\partial^2\phi}{\partial z^2}$,

$$\frac{\partial}{\partial \mathrm{T}} = \frac{\partial}{\partial t}, \qquad x'^2 + y'^2 + z'^2 = \mathrm{R}^2.$$

∴ we have

$$(\nabla^2\phi)_{xyzt} = (\nabla^2\phi)_{x'y'z't} + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \mp \frac{1}{c}\left\{(\phi)_{x'} + (\phi)_{y'} + (\phi)_{z'}\right\} \tag{6}$$

where

$$(\phi)_{x'} = \left\{\frac{\partial}{\partial x'}\left(\frac{x'}{\mathrm{R}}\,\frac{\partial\phi}{\partial \mathrm{T}}\right) + \frac{x'}{\mathrm{R}}\,\frac{\partial}{\partial \mathrm{T}}\left(\frac{\partial\phi}{\partial x'}\right)\right\}$$

and so on.

Also from (3) and (6) we have

$$-\lambda = (\Delta^2\phi)_{xyzt} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = (\nabla^2\phi)_{x'y'z't} \mp \frac{1}{c}\left\{(\phi)_{x'} + (\phi)_{y'} + (\phi)_{z'}\right\} . \tag{7}$$

We shall first evaluate ϕ at the origin in the (X'Y'Z', T) system and to do this, we surround the origin with a small sphere which shall form part of the boundary, the main boundary being elsewhere.

Now from Green's theorem, we have

$$\int \left[\phi \nabla^2\left(\frac{1}{R}\right)_{x'y'z'\,T} - \frac{1}{R}(\nabla^2 \phi)_{x'y'z',\,T}\right) dv' + \int \left[\frac{1}{R}\frac{\partial \phi}{\partial n} - \phi \frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right] d\sigma = 0$$

where $dv' = dx'\, dy'\, dz'$.

Since $\nabla^2\left(\frac{1}{R}\right) = 0$, we obtain from equation (7)

$$\int \frac{\lambda}{R} dv' \mp \frac{1}{c}\int \frac{\{(\phi)_{x'} + (\phi)_{y'} + (\phi)_{z'}\}}{R} dv' + \int \left(\frac{1}{R}\frac{\partial \phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{R}\right)\right) d\sigma = 0. \quad (8)$$

Now
$$\frac{(\phi)x'}{R} = \frac{1}{R}\left[\frac{1}{\partial x'}\left(\frac{x'}{R}\frac{\partial \phi}{\partial T}\right) + \frac{x'}{R}\frac{\partial}{\partial T}\left(\frac{\partial \phi}{\partial x'}\right)\right.$$
$$= \frac{1}{R}\left[\frac{\partial \phi}{\partial T}\frac{\partial}{\partial x'}\left(\frac{x'}{R}\right) + \frac{2x'}{R}\frac{\partial}{\partial T}\left(\frac{\partial \phi}{\partial x'}\right)\right]$$

also
$$\frac{\partial R}{\partial x'} = \frac{x'}{R}.$$

$$\therefore \quad \frac{(\phi)x'}{R} = \frac{\partial \phi}{\partial T}\left(\frac{3x'^2}{R^4} - \frac{1}{R^2}\right) + 2\frac{\partial}{\partial x'}\left(\frac{x'}{R^2}\frac{\partial \phi}{\partial T}\right)$$

with similar expressions for $\frac{(\phi)y'}{R}$ and $\frac{(\phi)z'}{R}$.

Since $x'^2 + y'^2 + z'^2 = R^2$, the second integral of (8) becomes

$$\frac{\partial \phi}{\partial T}\left(\frac{3R^2}{R^4} - \frac{3}{R^2}\right) + 2\left\{\frac{\partial}{\partial x'}\left(\frac{x'}{R^2}\frac{\partial \phi}{\partial T}\right) + \frac{\partial}{\partial y'}\left(\frac{y'}{R^2}\frac{\partial \phi}{\partial T}\right) + \frac{\partial}{\partial z'}\left(\frac{z'}{R^2}\frac{\partial \phi}{\partial T}\right)\right\}$$
$$= 2\left\{\frac{\partial}{\partial x'}\left(\frac{x'}{R^2}\frac{\partial \phi}{\partial T}\right) + \frac{\partial}{\partial y'}\left(\frac{y'}{R^2}\frac{\partial \phi}{\partial T}\right) + \frac{\partial}{\partial z'}\left(\frac{z'}{R^2}\frac{\partial \phi}{\partial T}\right)\right\}. \quad (9)$$

Substituting this in (8), and converting the surface integral, we have

$$\int \frac{\lambda}{R} dv' + \int \left\{\frac{1}{R}\frac{\partial \phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{R}\right) \mp \frac{2\cos\psi}{cR}\frac{\partial \phi}{\partial T}\right\} d\sigma \quad (10)$$

where ψ is the angle between the outward drawn normal to the surface and the radius vector from the origin at which the value of ϕ is sought.

Now in the contribution to the surface integral by the little sphere, we have in the limit zero from

$$\frac{1}{R}\frac{\partial \phi}{\partial n} \text{ and from } \mp \frac{2\cos\psi}{cR}\frac{\partial \phi}{\partial T} \text{ the term } -\phi\frac{\partial}{\partial n}\left(\frac{1}{R}\right) \longrightarrow -4\pi\phi.$$

Now with the surface integral applying to the main boundary, (10) becomes

$$4\pi\phi = \int \frac{\lambda}{R} dv' + \int \left\{\frac{1}{R}\frac{\partial \phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{R}\right) \mp \frac{2\cos\psi}{cR}\frac{\partial \phi}{\partial T}\right\} d\sigma.$$

Here the integrals are taken at constant T and $\frac{\partial}{\partial n}$ refers to differentiation along the normal in (x', y', z', T) system. Now changing to x, y, z, t, we have

$\frac{\partial}{\partial t} = \frac{\partial}{\partial T}$ and if l, m, n are the direction cosines of the outward-drawn normal to the surface

$$\frac{\partial \phi}{\partial n} = l\frac{\partial \phi}{\partial x'} + m\frac{\partial \phi}{\partial y'} + n\frac{\partial \phi}{\partial z'}.$$

From (4) we have

$$\begin{aligned}\frac{\partial \phi}{\partial n} &= l\frac{\partial \phi}{\partial x} + m\frac{\partial \phi}{\partial y} + n\frac{\partial \phi}{\partial z} \mp \frac{1}{c}\frac{\partial \phi}{\partial T}\left\{\frac{lx'}{R} + \frac{my'}{R} + \frac{nz'}{R}\right\} \\ &= \frac{\partial \phi}{\partial n} \pm \frac{\cos\psi}{cr}\frac{\partial \phi}{\partial t} \qquad (11)\end{aligned}$$

where now $\frac{\partial}{\partial n}$ refers to differentiation along the normal in (x,y,z,t) system.

Thus

$$4\pi\phi = \int \frac{[\lambda]}{r}dv + \int\left\{\frac{1}{r}\left[\frac{\partial \phi}{\partial n}\right] - [\phi]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \mp \frac{\cos\psi}{cr}\left[\frac{\partial \phi}{\partial t}\right]\right\}ds, \qquad (12)$$

which is the desired solution and where t is the time at which ϕ is sought at the origin. The understanding is that the quantities in the square brackets are to be evaluated at $t = \frac{r}{c}$ if minus sign is used or at $t = -\frac{r}{c}$ if the plus sign is used in (12).

ON EULER'S TOTIENT FUNCTION $\phi(n)$

BY

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The behaviour of $\dfrac{\phi(n+1)}{\phi(n)}$ as $n \longrightarrow \infty$, where $\phi(n)$ denotes the number of integers not greater than n and prime to it, does not seem to have been studied before. In this note the following results are proved.

$$(a) \quad \overline{\lim_{n\to\infty}} \frac{\phi(n+1)}{\phi(n)} = +\infty$$

$$(b) \quad \underline{\lim_{n\to\infty}} \frac{\phi(n+1)}{\phi(n)} = 0.$$

It will be obvious from the proofs given below that these results are true with $\phi(n+m)$ in the place of $\phi(n+1)$—one has only to take n_r in the form $\prod_{\substack{t\leqslant r \\ p_t \nmid m}} p_t$ in the first case and of the form $\prod_{\substack{t\leqslant r \\ p_t \nmid m}} p_t - m$ in the second case and in both cases for $r > m$.

Proof of (a).

Let $n_r = p_1\, p_2 \cdots p_r$ where p_t denotes the tth prime. Then every prime factor of $n_r + 1$ is obviously $> p_r$ and hence $n_r + 1 = \prod_{t=1}^{s} q_t^{\alpha_t}$ where the q's are distinct primes, α's are positive integers, $s < r$, and every q is greater than p_r. Hence follows

$$\prod_{t=1}^{s}\left(1-\frac{1}{q_t}\right) > \prod_{t=p_r+1}^{p_r+s}\left(1-\frac{1}{t}\right) > \tfrac{1}{2}.$$

Hence

$$\frac{\phi(n_r+1)}{\phi(n_r)} = \frac{(n_r+1)\prod_{t=1}^{s}\left(1-\frac{1}{q_t}\right)}{n_i \prod_{t=1}^{r}\left(1-\frac{1}{p_t}\right)}$$

$$> \tfrac{1}{2}\,\frac{n_r+1}{n_r}\left[\prod_{t=1}^{r}\left(1-\frac{1}{p_t}\right)\right]^{-1}$$

$$> \tfrac{1}{2}\,\frac{n_r+1}{n_r}\sum_{t=1}^{r}\frac{1}{t} \to \infty, \text{ as } r \to \infty.$$

Proof of (*b*).

Let $n_r = p_1 p_2 \cdots p_r$. Then every prime divisor of $n_r - 1$ is obviously $> p_r$ and therefore $n_r - 1 = \prod_{t=1}^{s} q_t^{a_t}$ where the q's are distinct primes, α's are positive integers, $s < r$ and every $q_s > p_r$. Hence follows

$$\prod_{t=1}^{s}\left(1-\frac{1}{q_t}\right) > \prod_{t=p_r+1}^{p_r+s}\left(1-\frac{1}{t}\right) > \tfrac{1}{2}.$$

Hence

$$\frac{\phi(n_r)}{\phi(n_r-1)} = \frac{n_r \prod_{t=1}^{r}\left(1-\frac{1}{p_t}\right)}{(n_r-1)\prod_{t=1}^{s}\left(1-\frac{1}{q_t}\right)}$$

$$< 2\,\frac{n_r}{n_r-1}\left[\prod_{t=1}^{r}\left(1-\frac{1}{p_t}\right)\right]$$

$$< 2\,\frac{n_r}{n_r-1}\,\frac{1}{\sum_{t=1}^{r}\frac{1}{t}} \to 0, \text{ as } r \to \infty.$$

ON RATIONAL (OR HERON) TRIANGLES

BY

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1. A *rational angle* is defined as one all of whose trigonometrical ratios are rational. It is evident that the angle θ is rational if and only if $\sin\theta$ and $\cos\theta$ are *both* rational, or if the *single* quantity $\tan\theta/2$ is rational. When θ and ϕ are rational, $\theta \pm \phi$ are rational. In particular, if θ is rational, so is 2θ and hence $2^n\theta$ where n is a positive integer. But $\theta/2^n$ will not ordinarily be rational.

2. We adopt the following usual notations in relation to a triangle ABC :

a, b, c = lengths of the sides; R = circum-radius; r, r_1, r_2, r_3 = in and ex-radii; p_1, p_2, p_3 = the lengths of the altitudes AD, BE, CF, O, S = the ortho-and circum-centres; I, I_1, I_2, I_3 = the in-and ex-centres; L, M, N = the intersections of AI, BI, CI, with BC, CA, AB; Δ = area of the triangle ABC.

A rational triangle is one in which a, b, c, Δ are rational. Since cos A, cos B, cos C and sin A, sin B, sin C can be expressed rationally in terms of a, b, c and Δ, it follows that *the angles A, B, C of a rational triangle are rational.* The quantities R, $r_1, r_2, r_3, p_1, p_2, p_3$, will also be rational for a rational triangle. Further if tan A/2, tan B/2 tan C/2 are respectively equal to u, v, w, which are three positive rational numbers satisfying the relation

$$uv + vw + wu = 1, \text{ i.e., } w = (1 - uv)/(u + v)$$

$$\text{then } a = 4\text{R}u/1 + u^2, \; b = 4\text{R}v/1 + v^2, \; c = 4\text{R}w/1 + w^2,$$

where R is an arbitrary rational number. The area of the triangle is given by

$$\Delta = 16\text{R}^2 uvw/(1 + u^2)(1 + v^2)(1 + w^2).$$

3. The four triangles $I_1I_2I_3$, $I\,I_2I_3$, $I\,I_3I_1$ and $I\,I_1I_2$ have ABC for their common pedal triangles. From the expressions for the sides and angles of these triangles it is clear that if one of them be rational, the others are also so. Also the sides and angles of the Δ $I_1I_2I_3$ are 4R cos A/2, 4R cos B/2, 4R cos C/2 and $90° - A/2$, $90° - B/2$, $90° - C/2$. If Δ $I_1I_2I_3$ be rational tan $(\pi - A)/4$, tan $(\pi - B)/4$, tan $(\pi - C)/4$ and R are rational and then the triangle ABC is also rational. The Δ ABC, whose four associated triangles, which have ΔABC for their com-

mon pedal triangle, are rational, will be called *absolutely rational* in this paper. If we put $u = \tan\frac{\pi - A}{4}$ $v = \tan\frac{\pi - B}{4}$ and $w = \tan\frac{\pi - C}{4}$ it is evident that u, v, w are positive rational numbers all less than unity and satisfying the relation

$$uv + wu + vw = 1.$$

Also because the values of sin A, sin B, sin C are given by

$$\sin A = 4u(1 - u^2)/(1 + u^2)^2, \text{ etc.},$$

the lengths of the three sides of an *absolutely rational* triangle can be exhibited in the form

$$a = 8Ru(1 - u^2)/(1 + u^2)^2,\ b = 8Rv(1 - v^2)/(1 + v^2)^2,\ c = 8Rw(1 - w^2)/(1 + w^2)^2$$

where R is an arbitrary rational number and u, v, w are rational numbers, which are all less than unity and satisfy the relation

$$uv + vw + wu = 1.$$

4. If S be the circum-centre of the rational triangle ABC, the three triangles SBC, SCA, SAB are all rational but the converse is not necessarily true. Similarly if O be the ortho-centre of a rational Δ ABC, the three triangles BOC, COA, AOB are all rational; moreover the pedal triangle DEF is also rational, although the rationality of this triangle (i.e. Δ DEF) does not necessarily imply that of the Δ ABC. In fact, Δ DEF is "absolutely rational" when Δ ABC happens to be rational. Also the rationality of any one of the triangles IBC, ICA, IAB implies the *absolute* rationality of Δ ABC. Similar conclusions hold good for the triangles I_1BC, I_1CA, I_1AB etc.

5. A quadrilateral is said to be rational when its sides, angles, and diagonals are rational. Assuming that the Δ ABC is absolutely rational, so that the Δ^s IBC, and I_1BC are rational, the sides and angles and the diagonal BC of the quadrilateral IBI_1C are rational. Now because from Ptolemy's Theorem, $IB.I_1C + IC.I_1B = BC.I\,I_1$, the diagonal $I\,I_1$ is also rational, and therefore the quadrilateral IBI_1C is rational. (The third diagonal I_2I_3 is also rational). Conversely if the quadrilateral IBI_1C is rational, because

$$I\,I_1 = \frac{IC}{\sin B/2} = \frac{IB}{\sin C/2} = \text{etc.}$$

and because BC($= a$) is rational, the triangle ABC is absolutely rational. Thus the rationality of the quadrilateral IBI_1C is the sure criterion for the triangle ABC being absolutely rational. The two other quadrilaterals ICI_2A, IAI_3B are also then rational.

6. If AL, BM, CN are the lengths of the bisectors of the angles of a rational triangle ABC, so that

$$AL = 2R \frac{\sin B \sin C}{\cos \frac{B-C}{2}}, \quad BM = 2R \frac{\sin C \sin A}{\cos \frac{C-A}{2}}, \quad CN = 2R \frac{\sin A \sin B}{\cos \frac{A-B}{2}},$$

the rationality of AL, BM, CN implies the rationality of A/2, B/2, C/2 (A, B, C, R being rational) so that ABC is absolutely rational.

Again taking into consideration the values of IA, IB, IC, I_1A, etc. for a rational triangle ABC, we are led to the conclusion that *the necessary and sufficient conditions for a rational* Δ *ABC to be absolutely rational are that any two lengths, which constitute the elements of the rectangular matrix* Ω *and which do NOT belong to the same row be rational where*

$$\Omega \equiv \begin{pmatrix} AL & IA & I_1A & I_2A & I_3A \\ BM & IB & I_1B & I_2B & I_3B \\ CN & IC & I_1C & I_2C & I_3C \end{pmatrix}.$$

Summarising the results, we find the conditions for a rational Δ *ABC to be absolutely rational, viz.*

(i) *any two of the angles A/2, B/2, C/2 are rational*

or (ii) *any one of the twelve associated triangles IBC, ICA, etc. is rational*

or (iii) *any one of the three quadrilaterals* IBI_1C, ICI_2A, IAI_3B *is rational*

or (iv) *any two elements of* Ω *not belonging to the same row are rational.*

7. If $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the pedal triangles of $\Delta, \Delta_1, \ldots, \Delta_{n-1}$, it is evident that the rationality of Δ implies the rationality of $\Delta_1, \ldots, \Delta_n$. Again there are 4 triangles which have Δ for their common pedal; of these, that which is acute angled will be called the first negative pedal of Δ and denoted by Δ_{-1} in the present paper. In this way we can define the *n*th negative pedal Δ_{-n} of Δ. If Δ_{-n} is rational, $\Delta_{-(n-1)}, \Delta_{-(n-2)}, \ldots, \Delta_{-1}, \Delta$ are all absolutely rational and in this case Δ is called a rational triangle of *rank n* and denoted by $\Delta^{(Rn)}$

Denoting by A_{-n}, B_{-n}, C_{-n}, the three angles of Δ_{-n}, we have

$$A_{-n} = \pi/2\left(1 - \frac{1}{2} + \frac{1}{2^2} - \cdots \text{ to } n \text{ terms}\right) + (-1)^n A/2^n$$

$$= \pi/3\left\{1 - \frac{(-1)^n}{2^n}\right\} + (-1)^n \frac{A}{2^n}.$$

Similarly $B_{-n} = \pi/3 \left\{1 - \frac{(-1)^n}{2^n}\right\} + (-1)^n \frac{B}{2^n}$,

and $C_{-n} = \pi/3 \left\{1 - \frac{(-1)^n}{2^n}\right\} + (-1)^n \frac{C}{2^n}$.

Further R_{-n}, the circum-radius of Δ_{-n}, is equal to $2^n R$. If therefore Δ_{-n} is rational, R also is rational and $\tan \frac{A_{-n}}{2}$ $(= u)$, $\tan \frac{B_{-n}}{2}$ $(= v)$, $\tan \frac{C_{-n}}{2} = (w)$ are rational and positive, connected by the relation

$$uv + vw + wu = 1.$$

Remembering that $\tan(2^n\theta)$ is a rational function $f(\tan\theta)$ of $\tan\theta$, we have,

$$f(u) = \tan(2^{n-1}A_{-n}) = \tan\left[(-1)^n A/2 + \frac{\pi}{6}\{2^n - (-1)^n\}\right],$$

$$f(v) = \tan\left[(-1)^n B/2 + \frac{\pi}{6}\{2^n - (-1)^n\}\right],$$

$$f(w) = \tan\left[(-1)^n C/2 + \frac{\pi}{6}\{2^n - (-1)^n\}\right].$$

Now $2^n - (-1)^n$ is always divisible by 3 but not by 2 and therefore $\frac{\pi}{6}\{2^n - (-1)^n\} = p\,\pi/2$ where p is an *odd* integer. Hence

$$f(u) = -\cot\{(-1)^n A/2\}, \quad f(v) = -\cot\{(-1)^n B/2\},$$
$$f(w) = -\cot\{(-1)^n C/2\}$$

i.e., $\tan A/2 = \pm 1/f(u)$, $\tan B/2 = \pm 1/f(v)$, $\tan C/2 = \pm 1/f(w)$

or, $\sin A = \frac{2|f(u)|}{1 + f^2(u)}$, $\sin B = \frac{2|f(v)|}{1 + f^2(v)}$, $\sin C = \frac{2|f(w)|}{1 + f^2(w)}$.

Hence the three sides of an unrestricted rational triangle $\Delta^{(Rn)}$ of rank n are representable in the form

$$a = \frac{4R|f(u)|}{1 + f^2(u)}, \quad b = \frac{4R|f(v)|}{1 + f^2(v)}, \quad c = \frac{4R|f(w)|}{1 + f^2(w)}$$

where R is an arbitrary rational number and u, v, w are three positive numbers satisfying $uv + vw + wu = 1$ and f denotes a uniquely determinate rational function.

UNION CURVES ON THE SURFACE OF REFERENCE OF A RECTILINEAR CONGRUENCE

BY

R. S. MISHRA, *Lucknow University*.

A *Union Curve* is a curve on the surface of reference of a Rectilinear Congruence, which has the property that its osculating plane at each point of the curve contains the line of the congruence through that point. The object of this paper is to find the expressions for the curvature and torsion of a union curve on the surface of reference of a Rectilinear Congruence. Some properties of these curves have been investigated.

1. Let a rectilinear Congruence be defined by the co-ordinates $x^i = x^i(u^1, u^2)$, $(i = 1, 2, 3)$ of a point P on the surface of reference S and by the direction cosines $\lambda^i = \lambda^i(u^1, u^2)$ of the line passing through P; it being assumed that the functions x^i and λ^i together with their partial derivatives up to the second order are continuous at P.

The functions λ^i may be expressed in terms of the direction numbers $x^i_{,\alpha}$ $(\alpha = 1, 2)$ of the tangents to the co-ordinate curves on the surface through P and the direction cosines X^i of the normal to the surface at P. Thus

$$\lambda^i = p^\alpha x^i_{,\alpha} + qX^i \tag{1·1}$$

where p^α are the contravariant components of a vector in the surface at P, q is a positive scalar function and $x^i_{,\alpha}$ denotes covariant differentiation of x^i with regard to u^α based on the first fundamental tensor

$$g_{\alpha\beta} = x^i_{,\alpha} x^i_{,\beta} \tag{1·2}$$

of the surface S.

If θ is the angle between the normal to S at P and the line λ of the congruence, it follows from (1·2), that

$$\cos\theta = \lambda^i \cdot X^i = q. \tag{1·3}$$

From (1·1) we get using (1·2)

$$\lambda^i \cdot x^i_{,\beta} = p^\alpha g_{\alpha\beta} = p_\beta \tag{1·4}$$

The angle ϕ between the tangent to a curve C : $u^\alpha = u^\alpha(s)$ through P on the surface and the line of the congruence through P with

direction λ^i is given by

$$\cos\phi = \lambda^i . \frac{dx^i}{ds} = \lambda^i \cdot x^i{}_{,\alpha} \cdot u'^{\alpha} = p_\alpha u'^{\alpha} \tag{1·5}$$

where dashes denote differentiation with regard to s.

2. Let C be a union curve and let the direction cosines of the tangent, principal normal and binormal to C at P be α^i, β^i, γ^i. The principal normal to C will lie in the plane of the lines with direction cosines λ^i and α^i $\left(= \frac{dx^i}{ds}\right)$ and hence

$$\beta^i = a \frac{dx^i}{ds} + b\lambda^i \tag{2·1}$$

where the functions a and b are obtained by multiplying (2·1) in turn by $\frac{dx^i}{ds}$ and β^i and using (1·5).

Thus $$a = -\cot\phi, \quad b = \frac{1}{\sin\phi}$$

and $$\beta^i = \operatorname{cosec}\phi\left(\lambda^i - \cos\phi \frac{dx^i}{ds}\right) \cdot \tag{2·2}$$

By Frenet's formulae

$$\frac{d\alpha^i}{ds} = \kappa\beta^i \tag{2·3}$$

where κ is the curvature of the curve C at x^i.

By virtue of (2·2), the equation (2·3) becomes

$$\frac{d^2x^i}{ds^2} = \kappa \operatorname{cosec}\phi\left(\lambda^i - \cos\phi \frac{dx^i}{ds}\right). \tag{2·4}$$

But as (Springer 1945)

$$\frac{d^2x^i}{ds^2} = \rho^\gamma \cdot x^i{}_{,\gamma} + \kappa_n X^i \tag{2·5}$$

where ρ^γ are the components of the curvature vector (Eisenhart, 1940, p. 187) and κ_n is the normal curvature of the surface at P in the direction of the curve, the equation (2·4) assumes the form

$$\rho^\gamma x^i{}_{,\gamma} + k_n X^i = \kappa \operatorname{cosec}\phi\left(\lambda^i - \cos\phi \frac{dx^i}{ds}\right). \tag{2·6}$$

Multiplying by X^i and summing for i we get,

$$\kappa_n = \kappa \cdot \operatorname{cosec}\phi \cdot q$$

in consequence of (1·3),

whence $$\kappa = \frac{\kappa_n}{q} \cdot \sin\phi. \tag{2·7}$$

Multiplying (2·6) by λ^i and summing for i, we get, by virtue of (1·3), (1·4) and (1·5)

$$\rho^\gamma \cdot p_\gamma + \kappa_n q = \kappa \sin \phi$$

or

$$\kappa = \operatorname{cosec} \phi \; (\rho^\gamma p_\gamma + \kappa_n q). \tag{2·8}$$

(2·7) and (2·8) are the two expressions for the curvature of a union curve.

From (2·7) and (2·8) we see that the normal curvature of the surface of reference in the direction of a union curve at the point where the ray cuts it, is given by

$$\kappa_n = \frac{p_\gamma \rho^\gamma \cos \theta}{\sin^2 \phi - \cos^2 \theta}. \tag{2·9}$$

If the congruence is formed by the normals to the surface of reference,

$$p_\alpha = 0, \; q = 1, \; \phi = \frac{\pi}{2}$$

and from (2·7) and (2·8)

$$\kappa = \kappa_n$$

i.e. for the congruence formed by normals to the surface of reference, the curvature of the union curve is the normal curvature of the surface in the direction of the curve. This is simply a consequence of the fact that the union curve in this case is a geodesic (Springer, 1945).

Other types of expressions for the curvature of the union curve can also be obtained from (2·6).

Multiplying this equation by $\frac{dX^i}{ds}$ $(= X^i{}_{,\alpha} u'^\alpha)$ and summing for i we get,

$$\rho^\gamma x^i{}_{,\gamma} \cdot X^i{}_{,\alpha} u'^\alpha = \kappa \operatorname{cosec} \phi \left(\lambda^i \cdot X^i{}_{,\alpha} u'^\alpha - \cos \phi \; x^i{}_{,\alpha} X^i{}_{,\beta} u'^\alpha u'^\beta\right) \tag{2·10}$$

or using (1·1) we get

$$-\sin \phi \cdot \rho^\gamma d_{\alpha\gamma} u'^\alpha = \kappa\left[\left(p^\beta x^i{}_{,\beta} + q X^i\right) X^i{}_{,\alpha} u'^\alpha + \cos \phi \cdot d_{\alpha\beta} u'^\alpha u'^\beta\right]$$

$$\text{or} \quad -\sin \phi \cdot \rho^\gamma d_{\alpha\gamma} u'^\alpha = \kappa\left[-p^\beta d_{\alpha\beta} u'^\alpha + p_\alpha u'^\alpha \kappa_n\right]$$

by virtue of (1·5).

Hence

$$\kappa = \frac{\sin \phi \cdot \rho^\gamma d_{\alpha\gamma} u'^\alpha}{p^\beta u'^\alpha \left(d_{\alpha\beta} - g_{\alpha\beta} \kappa_n\right)} \tag{2·11}$$

3. Torsion τ of the union curve is given by

$$\frac{d\gamma^i}{ds} = \tau \beta^i \tag{3·1}$$

where $\gamma^i = (\alpha^i \times \beta^i)$

Hence using (2·2), we get

$$\tau\left(\lambda^i - \cos\phi \cdot \frac{dx^i}{ds}\right) = -\cot\phi \cdot \frac{d\phi}{ds}\left(\frac{dx^i}{ds} \times \lambda^i\right) + \left(\frac{dx^i}{ds} \times \frac{d\lambda^i}{ds}\right) + \left(\frac{d^2x^i}{ds^2} \times \lambda^i\right) \tag{3·2}$$

Multiplying by λ^i, and summing for i we get

$$\tau \sin^2\phi = \left(\lambda^i \quad \frac{dx^i}{ds} \quad \frac{d\lambda^i}{ds}\right)$$

or

$$\tau = \operatorname{cosec}^2\phi\left(\lambda^i \quad x^i_{,\alpha} \quad \lambda^i_{,\beta}\right) u'^\alpha u'^\beta$$

or

$$\tau = -\operatorname{cosec}^2\phi \cdot \xi_{\alpha\beta} u'^\alpha u'^\beta \tag{3·3}$$

where $\xi_{\alpha\beta}$ are the coefficients of Sannia's second quadratic form [Behari and Mishra 1949, p. 91–92, § (7)].

If d is the parameter of distribution of a ruled surface through a line of a Rectilinear congruence

$$d = \xi_{\alpha\beta} \frac{du^\alpha}{d\sigma} \cdot \frac{du^\beta}{d\sigma} \tag{3·4}$$

where $d\sigma$ is a small element of length on the spherical representation of the congruence.

By virtue of (3·4), the equation (3·3) assumes the form

$$\tau = -\operatorname{cosec}^2\phi \cdot d \cdot \left(\frac{d\sigma}{ds}\right)^2 \tag{3·5}$$

Some other expressions for τ can be obtained from (3·2), by scalar multiplication of (3·2) by X^i and $\frac{d\lambda^i}{ds}$.

Multiplication of (3·2) with $\frac{dx^i}{ds}$, and use of (1·5) yields

$$\left(\frac{dx}{ds^i} \quad \frac{d^2x^i}{ds^2} \quad \lambda^i\right) = 0.$$

or

$$\left(x'_{,\alpha} \quad p^\gamma x^i_{,\gamma} + \kappa_n X^i \quad p^\beta x^i_{,\beta} + q X^i\right) u'^\alpha = 0$$

by virtue of (1·1(and (2·5)

or

$$e_{\alpha\beta}\left(p^\beta q - p^\beta \cdot \kappa_n\right) u'^\alpha = 0 \tag{3·6}$$

which is the equation of a union curve (Springer, 1945) obtained in a much simpler way.

My thanks are due to Prof. C. N. Srinivasiengar for suggesting modifications in the paper, after its submission for publication.

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NOTES AND DISCUSSIONS

On The Equation $x_1^3 + x_2^3 = y_1^3 + y_2^3$

II

In a previous paper[1] of the same title, I have obtained the following theorems:

Theorem A. If a_1, a_2, b_1, b_2 be numbers satisfying the relations,

$$a_1^3 + a_2^3 + b_1^3 + b_2^3 = 0, \tag{1}$$

$$-(a_1 + b_1)(a_2 + b_2) \text{ is a perfect square,} \tag{2}$$

then a solution of the equation

$$x_1^3 + x_2^3 = y_1^3 + y_2^3 \tag{3}$$

is given by

$$\left.\begin{array}{ll} x_1 = a_1x^2 + \alpha_1xy - b_1y^2, & x_2 = a_2x^2 + \alpha_2xy - b_2y^2 \\ y_1 = a_1y^2 + \alpha_1xy - b_1x^2, & y_2 = a_2y^2 + \alpha_2xy - b_2x^2 \end{array}\right\} \tag{4}$$

where

$$\left.\begin{array}{l} \alpha_1 = \varepsilon\,(a_2 - b_2)\sqrt{-(a_2 + b_2)(a_1 + b_1)} \\ \alpha_2 = \varepsilon\,(a_1 - b_1)\sqrt{-(a_1 + b_1)(a_2 + b_2)} \\ \varepsilon = \pm 1 \end{array}\right\}. \tag{5}$$

An interesting special case of Theorem A is

Theorem B. If a_1, b_1, a_2, b_2 be numbers satisfying (1) and

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = 1, \tag{6}$$

then a solution of (3) is given by

$$\left.\begin{array}{ll} x_1 = a_1x^2 + \varepsilon a_1xy - b_1y^2, & x_2 = a_2x^2 + \varepsilon a_2xy - b_2y^2 \\ y_1 = a_1y^2 + \varepsilon a_1xy - b_1x^2, & y_2 = a_2y^2 + \varepsilon a_2xy - b_2x^2 \\ \varepsilon = \pm 1 & \end{array}\right\}. \tag{7}$$

In this paper it is proposed to find solutions of (1) satisfying conditions (2) or (6).

It is well known that the general solution of (1) is given by

$$\left.\begin{array}{ll} a_1 = (a + 3b)(a^2 + 3b^2) - 1, & a_2 = (a^2 + 3b^2)^2 - (a + 3b) \\ b_1 = -(a - 3b)(a^2 + 3b^2) + 1, & b_2 = (a - 3b) - (a^2 + 3b^2)^2 \end{array}\right\}. \tag{8}$$

If these satisfy (6), we have on simplication,

$$(a - b)\,\{(a^2 + 3b^2)^3 - 2a\,(a^2 + 3b^2) + 1\} = 0.$$

It is easy to see that the second factor does not vanish for rational values of a and b. Therefore we have

$$a = b,$$

and the solution (8) reduces to

$$a_1 = 16a^3 - 1, \qquad a_2 = 16a^4 - 4a$$
$$b_1 = 8a^3 + 1, \qquad b_2 = -(2a + 16a^4),$$

Replacing $2a$ by a, we get finally

Theorem 1. A common solution of (1) and (6) is given by

$$\left.\begin{array}{ll} a_1 = 2a^3 - 1, & a_2 = a^4 - 2a \\ b_1 = a^3 + 1, & b_2 = -a - a^4 \end{array}\right\} . \tag{9}$$

From (8), we have on simplication.

$$-(a_1 + b_1)(a_2 + b_2) = (6b)^2 (a^2 + 3b^2).$$

The right side will be a perfect square if and only if

$$a^2 + 3b^2 = c^2 \tag{10}$$

The general solution of (10) is easily found to be

$$a = f(3p^2 - q^2), \quad b = 2fpq, \quad c = f(3p^2 + q^2) \tag{11}$$

Substituting from (11) in (8), we get a solution of (1) and (2), as

$$\left.\begin{array}{l} a_1 = f^3 (3p^2 - q^2 + 6pq)(3p^2 + q^2)^2 - 1 \\ b_1 = f^3 (6pq - 3p^2 + q^2)(3p^2 + q^2)^2 + 1 \\ a_2 = f^4 (3p^2 + q^2)^4 - f(3p^2 - q^2 + 6pq) \\ b_2 = f(3p^2 - q^2 - 6pq) - f^4 (3p^2 + q^2)^4 \end{array}\right\} . \tag{12}$$

Reference

[1] *Math. Student*—Vol. XIII, No. 2. June 1945.

P. KESAVA MENON.

On Kemmer's Identity in Combinatory Functions

1. *Notations.*

$$\binom{m}{p} = \frac{m(m-1)(m-2)\cdots p \text{ factors}}{p\,!}, \text{ where } m \text{ is any number}$$

and p is a positive integer.

$$\binom{m}{p} = 1, \text{ if } p = 0, \binom{m}{p} = 0 \text{ if } p \text{ is negative.}$$

and $\binom{n}{c}_r = n(n+c)(n+2c)\cdots r$ factors ; $\binom{n}{c}_r = 1$ if r is zero or negative.

2. *Introduction.*

Kemmer obtained the identity[1]

$$\binom{2s}{s \pm k} = \sum_{0 \leq p \leq \frac{1}{2}(s-k)} 2^{s-2p-k} \binom{s}{p} \binom{s-p}{p+k} \tag{1}$$

and proved it with the help of elementary properties of Γ functions and the hypergeometric function ${}_2F_1$.. The identity has attracted the attention of some Indian mathematicans who established the identity by elementary methods[2] and obtained some generalizations of the same. I shall point out here that Kemmer's identity is a particular case of an identity [3] which I published in 1922. This identity is as follows :— .

$$\binom{x}{1}_n = \sum_{0 \leq p \leq n} \binom{\delta + n - 1}{-1}_p \binom{x - \delta}{-1}_{n-p} \binom{n}{p}. \tag{2}$$

[Here if n is a fraction, we are to take the integral part of n for n]

3. The theorem (1) may be written

$$\binom{2m}{2n} = \sum_{0 \leq p \leq n} 2^{2n-2p} \binom{m}{p} \binom{m-p}{2n-2p} \tag{3}$$

where $2n$ is an even or odd integer. It is to be noted that n in (2) and (3) represents the same number.

4. *Case* (i) *when $2n$ is an even integer.* Let us consider the particular case when $2n = 6$. If we substitute $\delta = \frac{1}{2}$ and $2x = 2m - 5$ in (2) and then multiply both sides by $\frac{2m\,(2m-2)\,(2m-4)}{6\,!}$, we have

$$\binom{2m}{6} = \sum_{0 \leq p \leq 3} 2^{6-2p} \binom{m}{p} \binom{m-p}{6-2p}.$$

The general case may be similarly treated.

Case (ii) *When $2n$ is an odd integer.* Let $2n = 5$, then (2) becomes

$$\binom{x}{1}_2 = \sum_{0 \leq p \leq 2} \binom{\delta + 1}{-1}_p \binom{x - \delta}{-1}_{2-p} \binom{2}{p}. \tag{4}$$

If we put $\delta = \frac{1}{2}$ and $2x = 2m - 3$ in (4) and then multiply both sides by $\frac{2m\,(2m-2)\,(2m-4)}{5\,!}$, we have

$$\binom{2m}{5} = \sum_{0 \leq p \geq \frac{5}{2}} 2^{5-2p} \binom{m}{p} \binom{m-p}{5-2p}.$$

The general case may be similarly shown.

Thus if $\delta = \frac{1}{2}$ or $\frac{3}{2}$ according as $2n$ is even or odd, (2) reduces to (3). Besides n which is the same positive integer in both (2) and (3), there are two arbitrary constants x and δ in (2), while there is only one arbitrary constant m in (3). So (3) or (1) is a particular case of (2).

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The Solar Compass

From an observation of the altitude of a known celestial body at a given station, the method of calculating its azimuth is well-known in Spherical Astronomy. To translate this theoretical process into a practical method whereby the north point of the compass is determined easily and quickly, is the object of the Solar compass chart described below.

If a vertical style of definite height is placed at a fixed point O on a paper spread on a horizontal plane surface, the locus of the sun's shadow of the top point of the style in the course of a day is a plane section of a right circular cone whose axis points to the north pole and whose semi-vertical angle is equal to the complement of the solar declination. It is a hyperbola reducing to a straight line when the sun is at the equinoxes. Assuming the same position for the paper and the style, it is possible by elementary geometrical methods described below to draw the shadow curves for all the days of the year, or, for the sake of convenience the curves corresponding to half degree intervals in the sun's declination.

To use the chart so constructed, place it on a horizontal plane surface and the vertical style at O. Next adjust the position of the chart so that the shadow point of the top of the style falls on the curve corresponding to the date of observation. Then the common axis of the hyperbolas indicates the north south direction.

The accuracy obtained by using the chart varies from ½° at 5 P.M. to 1° at 3 P.M. and 4° at 1 P.M., the method totally failing when the sun is in the meridian.

To determine geometrically the shadow point when the sun's declination and the hour angle are given. Let O be the foot and P the top point of the vertical style OP. Draw a straight line ON on the paper so that angle ONP = latitude. Produce NO to T so that angle NPT = 1 rt. angle. Draw TR perp. to NT so that TPR = hour angle of the sun. Then if S is any point on NR, it is the shadow point when the declination is SPR.

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SOME INTERESTING PROBLEMS & THEIR SOLUTIONS

BY

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I. *Problem.* (M_n) is a steadily increasing positive sequence tending to infinity. It is known that

$$f(p) = \sum_{n=1}^{\infty} \frac{M_{n+1} - M_n}{M_{n+1} \; M_n^p}$$

converges for $p > 0$. Prove that, as $p \to +0$,

$$\text{(i)} \quad p f(p) \to 1 \text{ when } \frac{M_{n+1}}{M_n} \to 1 \text{ as } n \to \infty;$$

$$\text{(ii)} \quad p f(p) \to \left(1 - \frac{1}{c}\right) \frac{1}{\log c} \text{ when } \frac{M_{n+1}}{M_n} \to c > 1:$$

$$\text{(iii)} \quad p f(p) \to 0 \quad \text{when } \frac{M_{n+1}}{M_n} \to \infty.$$

[See, Bromwich: Infinite Series, (1909 Ed.), p. 44, ex. 12]

Solution. (i) $\dfrac{M_{n+1}}{M_n} \to 1$ as $n \to \infty$. In this case

$$f_1(p) = \sum_{n=1}^{\infty} \frac{M_{n+1} - M_n}{M_{n+1}^{1+p}} \text{ and } f_2(p) = \sum_{n=1}^{\infty} \frac{M_{n+1} - M_n}{M_n^{1+p}}$$

both converge for $p > 0$ and

$$f_1(p) \leqslant f(p) \leqslant f_2(p). \tag{1}$$

Given $\varepsilon > 0$, we can find n_0 so that $1 \leqslant \dfrac{M_{n+1}}{M_n} \leqslant 1 + \varepsilon$ for $n \geqslant n_0$. Now, as $p \to +0$,

$$p f_2(p) = p \sum_{1}^{n_0 - 1} + p \sum_{n_0}^{\infty} \leqslant O(p) + p(1+\varepsilon)^{1+p} \sum_{n_0}^{\infty} \frac{M_{n+1} - M_n}{M_{n+1}^{1+p}}$$

$$\leqslant O(p) + p(1+\varepsilon)^{1+p} \int_{M_{n_0}}^{\infty} \frac{dx}{x^{1+p}}$$

$$= O(p) + \frac{(1+\varepsilon)^{1+p}}{M^p_{n_0}}.$$

So, by (1), $\overline{\lim\limits_{p \to +0}} \; p f(p) \leqslant \overline{\lim\limits_{p \to +0}} \; p f_2(p) \leqslant 1 + \varepsilon$. Since $\varepsilon > 0$ is as small as we like, we get that $\overline{\lim\limits_{p \to +0}} \; p f(p) \leqslant 1$. Similarly

$$p f_1(p) = p \sum_{1}^{n_0-1} + p \sum_{n_0}^{\infty} \geqslant O(p) + \frac{p}{(1+\varepsilon)^{1+p}} \sum_{n_0}^{\infty} \frac{M_{n+1} - M_n}{M_n^{1+p}}$$

$$\geqslant O(p) + \frac{p}{(1+\varepsilon)^{1+p}} \int_{M_{n_0}}^{\infty} \frac{dx}{x^{1+p}}$$

$$= O(p) + \frac{1}{M^p_{n_0}(1+\varepsilon)^{1+p}}.$$

Hence, by (1), $\underline{\lim}_{p \to +0} p f(p) \geqslant \underline{\lim}_{p \to \infty} p f_1(p) \geqslant \frac{1}{1+\varepsilon}$ and as before this gives $\underline{\lim}_{p \to +0} p f(p) \geqslant 1$. Therefore we get along with the previous result that $p f(p) \to 1$ as $p \to +0$.

(ii) $\frac{M_{n+1}}{M_n} \to c > 1$ as $n \to \infty$. Here let $\varepsilon > 0$ be such that $c - \varepsilon > 1$. Find n_0 so that $c - \varepsilon \leqslant \frac{M_{n+1}}{M_n} \leqslant c + \varepsilon$ for $n \geqslant n_0$.

Then we get

$$M_{n_0}(c-\varepsilon)^{n+1-n_0} \leqslant M_{n+1} \leqslant M_{n_0}(c+\varepsilon)^{n+1-n_0},$$

which could be written as

$$A(\varepsilon)(c-\varepsilon)^{n+1} \leqslant M_{n+1} \leqslant B(\varepsilon)(c+\varepsilon)^{n+1}, \; n \geqslant n_0$$

where $A(\varepsilon) = M_{n_0}(c-\varepsilon)^{-n_0}$ and $B(\varepsilon) = M_{n_0}(c+\varepsilon)^{-n_0}$. Now, as $p \to +0$,

$$p f(p) = p \sum_{1}^{n_0} + p \sum_{n_0+1}^{\infty} \leqslant O(p) + p \sum_{n_0+1}^{\infty} \left(1 - \frac{1}{c+\varepsilon}\right) [A(\varepsilon)]^{-p} (c-\varepsilon)^{-np}$$

$$= O(p) + p\left(1 - \frac{1}{c+\varepsilon}\right) [A(\varepsilon)]^{-p} (c-\varepsilon)^{-p(n_0+1)} \left[1 - \frac{1}{(c-\varepsilon)^p}\right]^{-1} \qquad (2)$$

Hence $\overline{\lim}_{p \to +0} p f(p) \leqslant \left(1 - \frac{1}{c+\varepsilon}\right) \frac{1}{\log (c-\varepsilon)}$ and since $\varepsilon > 0$ is arbitrarily small, we get

$$\overline{\lim}_{p \to +0} p f(p) \leqslant \left(1 - \frac{1}{c}\right) \frac{1}{\log c} \qquad (3)$$

Similarly

$$p f(p) \geqslant O(p) + p\left(1 - \frac{1}{c-\varepsilon}\right) [B(\varepsilon)]^{-p} (c+\varepsilon)^{-p(n_0+1)} \left[1 - \frac{1}{(c+\varepsilon)^p}\right]^{-1}$$

which gives, as before, that $\underline{\lim}_{p \to +0} p f(p) \geqslant \left(1 - \frac{1}{c}\right) \frac{1}{\log c}$. So $p f(p) \to \left(1 - \frac{1}{c}\right) \frac{1}{\log c}$ as $p \to +0$.

(iii) Since $\frac{M_{n+1}}{M_n} \to \infty$, the relation (2) and therefore (3) will be valid however large $c > 1$ be. Hence $\overline{\lim_{p \to +0}}\ p f(p) \leqslant 0$ which gives $p f(p) \to 0$ as $p \to +0$.

II. *Problem.* Let $a > 0$ and (b_n) be a bounded sequence. Let $M_n = an + b_n$ and let no M_n be zero. Let

$$f(p) = \sum_{1}^{\infty} \frac{a^{1+p}}{M_n^{1+p}}, \quad p > 0.$$

Prove that $f(p) - 1/p$ tends to a finite limit as $p \to +0$.

Solution. We may suppose without loss of generality that $a = 1$.

$$\text{Let } u_n(p) = \frac{1}{M_n^{1+p}} - \int_n^{n+1} \frac{dx}{x^{1+p}}. \quad \text{Then}$$

$$f(p) - \frac{1}{p} = \sum_{n=1}^{\infty} u_n(p) \text{ for } p > 0$$

$$\text{Now} \quad u_n(p) = \int_n^{n+1} \left[\frac{1}{(n+b_n)^{1+p}} - \frac{1}{x^{1+p}}\right] dx$$

$$= + \int_n^{n+1} \frac{(1+p)\,(x-n-b_n)}{[\,x + \theta\,(x-n-b_n)\,]^{2+p}}\, dx, \quad 0 < \theta < 1$$

$$= O\left(\frac{1}{n^{1+\delta}}\right)$$

for all $p \geqslant \delta - 1$ where $\delta > 0$, the constant in the O depending on δ only. Hence $\sum u_n(p)$ is uniformly convergent for $p \geqslant \delta - 1$ and each $u_n(p)$ is continuous for $p \geqslant \delta - 1$. So $\sum u_n(p) = \phi(p)$ is defined and continuous for $p \geqslant \delta - 1$ and $\phi(p) = f(p) - 1/p$ for $p > 0$.

Hence $f(p) - 1/p = \phi(p) \to \phi(0)$ as $p \to +0$.

As a special case, let $b_n = 0$ and $a = 1$. We see that

$$\sum_{1}^{\infty} \frac{1}{n^{1+p}} - \frac{1}{p} \text{ tends to } \phi(0) = \sum_{1}^{\infty} \left\{\frac{1}{n} - \log\left(1 + \frac{1}{n}\right)\right\}$$

which is equal to the Euler's constant γ as $p \to +0$.

The Mean Deviation Locus

The chief property of the mean deviation is that it is a minimum when measured from the median. Unfortunately, no complete proof of this is available in text-books, covering all possibilities. H. Cramer[1] just suggests a proof based on two identities and in a footnote indicates how modifications have to be made for discontinuities. A. C. Aitken[2] goes into the proof of the minimal property in greater detail but omits the case of repeated discontinuous variates. In a recent issue of the *Math. Student*,[3] Seetharaman attempts to prove this omitted case but does not distinguish properly between the determinate and the indeterminate median. We propose to set right these defects and point out further a mean deviation locus which clarifies and accounts for the minimal property. Incidentally, we indicate a few algebraical identities helpful in the computation of the mean deviation.

2. We start with two fundamental identities for the general discrete case.

Let D_i denote the mean deviation of a distribution of n variates about the value i and f_i, g_i the total number of variates greater than i and less than i respectively. Then we have

$$\left.\begin{array}{ll}\text{(i)} & \text{if } i > j,\ n(D_j - D_i) = (2f_j - n)(i-j) - 2\sum\limits_{j+}^{i-} |x-i|\,\phi(x) \\ \text{(ii)} & \text{if } i < j,\ n(D_j - D_i) = (2g_j - n)(j-i) - 2\sum\limits_{i+}^{j-} |x-i|\,\phi(x)\end{array}\right\} \qquad \text{I}$$

where $\phi(x)$ is the frequency of the variate x.

To prove I (i), we have only to consider separately the absolute deviations about i and j of the variates in the open interval $I(j, i)$ or (i, j) and its right and left adjoining intervals, say R and L.

When $i > j$, the contributions to the left hand side of I (i) from the intervals I, R, L are respectively

$$\left.\begin{array}{l}\sum\limits_{j+}^{i-} (2x - 2i + i - j)\,\phi(x), \\ \{f_i + \phi(i)\}\,(i-j) \\ \{g_j + \phi(j)\}\,(j-i)\end{array}\right\} \qquad \text{II}$$

and (the last of these)

whence the result follows by addition, if we remember that

$$\sum_{j+}^{i-} \phi(x) + \phi(i) + f_i = f_j \text{ and } f_j + \phi(j) + g_j = n. \qquad \text{III}$$

The identity I (ii) is proved similarly.

We may note now that the right-hand sides of both I (i) and I (ii) cannot be positive when $f_j \leqslant \frac{n}{2}$ and $g_j \leqslant \frac{n}{2}$, which gives rise to *four* cases. In all these cases $D \leqslant D_i$ where i has any value.

When $f_j < \frac{n}{2}$ and $g < \frac{n}{2}$, we have $\phi(j) = n - f_j - g > 0$ and j belongs to the distribution and is the unique determinate median thereof, since $f_j + \phi(j)$ and $g_j + \phi(j)$ are both greater than $\frac{n}{2}$. When $f = \frac{n}{2} = g_j$ or $f_j = \frac{n}{2} > g_j$ or $f_j < \frac{n}{2} = g_j$, j is the indeterminate median being any point in the open interval (k, l) or k or l, where $f_k = \frac{n}{2} = g_l$ and the corresponding open interval does not contain any variate of the distribution while k and l belong to the distribution.

Thus, when j is a median, D_j is the least mean deviation.

3. Let a, b, $\bar{x}$ be the lowest value, the highest value and the arithmetic mean respectively of the distribution. Obviously $D_a = \bar{x} - a$, $D_b = b - \bar{x}$, while if u is any value less than a and v any value greater than b, we have

$$\left.\begin{aligned} D_u - D_a &= a - u, \\ \text{and} \quad D_v - D_b &= v - b. \end{aligned}\right\} \qquad \text{III}$$

From our fundamental identity I, if $p > x > q$,

$$\text{then} \qquad D_x - D_q = \frac{2f_q - n}{n}(q - x) \qquad \text{IV}$$

when the terms under the sigma are nul and p, q are two consecutive variates of the distribution.

We may now define the mean deviation locus as the locus of the point (x, D_x) for all real values of x.

From III and IV we infer that this locus consists of a finite set of straight-line-segments whose slopes range from -1 to 1. If j is the lowest value of the variates for which $f_j < \frac{n}{2}$, the slope given by $(n - 2\bar{f_q})/n$ changes from a negative value to a positive value passing through the value zero or abruptly according as g_j is equal to $\frac{n}{2}$ or less than $\frac{n}{2}$. Thus D_q diminishes up to $q = j$ and then

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increases. When $f_j < \frac{n}{2}$ and $g_j = \frac{n}{2}$, there is a value k just below j in the distribution for which $f_k = \frac{n}{2}$ and from (IV) we have $D_x = D_k$ for $j \geqslant x \geqslant k$.

When the variate is continuous, the mean deviation locus becomes a continuous curve convex to the x-axis and having its minimum point at (j, D_j) where j is the median.

For, $\frac{D_i - D_j}{i - j} = \frac{n - 2f}{n}$ becomes in the limit when $i, j \longrightarrow x$.

$$\frac{d}{dx}(D_x) = 1 - \frac{2\int_x^{\infty} \phi(x)\,dx}{\int_{-\infty}^{+\infty} \phi(x)\,dx}$$

where $\phi(x)$ is the probability density of x or

$$\frac{d^2}{dx^2}(D_x) = 2\phi(x).$$

The minimum value of D_x occurs when $\frac{d}{dx}(D_x) = 0$, i.e.

$\int_x^{\infty} \phi(x)\,dx = ½ \int_{-\infty}^{\infty} \phi(x)\,dx = ½$ i.e. when x is the median,

and $\frac{d^2}{dx^2}(D_x)$ is always positive; the curve is convex to the x-axis.

Thus the theorem is also proved for the continuous case.

4. Lastly, changing the notation slightly, let us denote by D_r the mean deviation about the rth variate (x_r) with frequency ϕ_r, the variates being in ascending order of magnitude from the lowest value x_1. From our fundamental identity, we have

$$D_r - D_{r-1} = (x_r - x_{r-1})\left(1 - \frac{2f_{r-1}}{n}\right) \text{ where } f_{r-1} = \phi_r + \phi_{r+1} + \cdots$$

Hence $\sum_{r=2}^{r=i}(D_r - D_{r-1}) = \sum_2^i (x_r - x_{r-1}) - \frac{2}{n}\sum_2^i (x_r - x_{r-1})f_{r-1}$

i.e. $D_i - D_1 = x_i - x_1 - \frac{2}{n}\sum_2^i (x_r - x_{r-1})\,f_{r-1}$

But $D_1 = \bar{x} - x_1$, so that

$$D_i = \bar{x} + x_i - 2x_1 - \frac{2}{n}\sum_2^i (x_r - x_{r-1})\,f_{r-1}; \tag{5}$$

the terms in the summation on the right-hand side can be easily calculated from a table of differences of variates and cumulative sums of frequencies from the upper end, i.e. the frequency associated with the highest value.

Cor.: $D_i < \bar{x}$ if $x_i \leqslant 2x_1$.

References

1. H. CRAMER: *Mathematical Methods of Statistics*, 1946, p. 179,
2. A. C. AITKEN: *Statistical Mathematics*, 1945. (pp. 32—34).
3. V. SEETHARAMAN: *Math. Student*, XV, (1 & 2), 50.

Postscript: Mr. Thawari's proof in *Math. Student*, XVI (1 & 2) 35, 36 does not discuss adequately the discontinuous case. Stieltjes integrals are apparently used on the same terms as ordinary integrals.

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The Volume of a tetrahedron

This short note contains a *simple* solution of the following familiar result.

The volume of a tetrahedron is $\frac{1}{6}$ $abd \sin \theta$, where a and b are the lengths of a pair of opposite edges, θ is the angle and d is the S.D. (shortest distance) between them. For the sake of brevity, the notation (ABCD) has been used to denote the volume of the tetrahedron, formed by the four non-coplanar points A,B,C,D.

Suppose that in a given tetrahedron AB ($\equiv a$) and CD ($\equiv b$) are a pair of opposite edges, located respectively along the lines L and M and that PR ($\equiv d$) is the shortest distance (S.D.) between them.

If the points C,D be held *fixed* and the edge AB be made to *slide* (along its own line L) into a new position A'B' (so that, as a matter of course, A'B' = AB = a), it is plain that the two triangles CAB, CA'B' are equal in area, for they are situated in a common plane π (passing through the point C and the line L) and have the *same* vertex and stand on *equal* bases. Further, if p denotes the perpendicular dropped from D on the plane π, we must have

$$(ABCD) = \tfrac{1}{3}\, p \times \triangle CAB \quad \text{and} \quad (A'B'CD) = \tfrac{1}{3}\, p \times \triangle CA'B',$$

so that
$$(ABCD) = (A'B'CD). \tag{1}$$

Parity of reasoning shows that if the edge A′B′ be now kept *fixed* and the edge CD slides into a new position C′D′ (along M), so that C′D′ = CD = b, we must have :

$$(A'B'CD) = (A'B'C'D') \tag{2}$$

Thus (1) and (2) lead to (ABCD) = (A′B′C′D′), signifying that the volume of a tetrahedron remains *invariant* in whichever way two opposite edges slide along their own lines (without any alteration in their lengths).

If then we cut off PQ = a and RS = b from (L) and (M) respectively where PR is the S.D. between L and M, and imagine the edges AB and CD to *slide* along their own lines into the respective positions PQ and RS, we obtain :

$$(ABCD) = (PQRS) = \tfrac{1}{3}\ \triangle RPQ \times \text{(perpendicular from S on plane PQR)}$$
$$= \tfrac{1}{3} \times (\tfrac{1}{2} PQ \times PR) \times (RS\ \sin\theta).$$

for (Euclid, XI) the line, joining R to the foot (T) of the perpendicular dropped from S on the plane PQR is parallel to PQ (or L).

Thus we have

$$(ABCD) = \tfrac{1}{6}\, abd \sin\theta.$$

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Areal Co-ordinates Treated Vectorially

1. Let P be any point in the plane of a △ ABC, and let AP, BP, CP (produced if necessary) meet BC, CA, AB in A′, B′, C′. Since P, A, B, C are coplanar, there exists a linear relation $a\,\overline{PA} + b\,\overline{PB} + c\,\overline{PC} = 0$. Let $\overline{PA'} = x\cdot PA$, $\overline{PB'} = y\cdot PB$, $\overline{PC'} = z\cdot\overline{PC}$. Hence $\frac{a}{x}\cdot\overline{PA'} + b\cdot PB + c\cdot PC = 0$. Since A′, B, C are collinear, we have

$$\frac{a}{x} + b + c = 0, \text{ or } x = -\frac{a}{b+c}\cdot \quad \text{Similarly } y = -\frac{b}{c+a},\ z = \frac{-c}{a+b}.$$

We have

$$\frac{\triangle PBC}{\triangle ABC} = \frac{PA'}{AA'} = \frac{x\cdot PA}{(x-1)PA} = \frac{a}{a+b+c},$$

and similar expressions involving △ PCA, △ PAB.

Introducing any origin O, the relation $a\cdot\overline{PA} + b\cdot\overline{PB} + c\cdot\overline{PC} = 0$ gives

$$a(\overline{A} - \overline{P}) + b(\overline{B} - \overline{P}) + c(\overline{C} - \overline{P}) = 0.$$

We thus obtain

$$\boldsymbol{p} = \mathrm{X}\boldsymbol{p}_1 + \mathrm{Y}\boldsymbol{p}_2 + \mathrm{Z}\boldsymbol{p}_3 \qquad (1)$$

where X, Y, Z are the areal coordinates of P, and $\boldsymbol{p}$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$ are the position vectors of the points P, A, B, C respectively.

2. Let Ox, Oy be any pair of Cartesian coordinate axes (rectangular or oblique), and $\boldsymbol{i}$, $\boldsymbol{j}$ unit vectors along them. If the coordinates of P, A, B, C be (x, y), (x_1, y_1), (x_2, y_2) and (x_3, y_3), we have $\boldsymbol{p} = \boldsymbol{i}x + \boldsymbol{j}y$, etc. Substituting in (1), we obtain

$$x = \mathrm{X}x_1 + \mathrm{Y}x_2 + \mathrm{Z}x_3$$
$$y = \mathrm{X}y_1 + \mathrm{Y}y_2 + \mathrm{Z}y_3,$$

the transformation formulae from cartesian to areals.

3. Let $(\mathrm{X}_i, \mathrm{Y}_i, \mathrm{Z}_i)$, $i = 1, 2, 3$ be the areal coordinates of three points P_1, P_2, P_3. Their position vectors are $\boldsymbol{r}_i = \mathrm{X}_i\boldsymbol{p}_1 + \mathrm{Y}_i\boldsymbol{p}_2 + \mathrm{Z}_i\boldsymbol{p}_3$.

Hence the scalar triple product

$$[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3] = \begin{vmatrix} \mathrm{X}_1 & \mathrm{Y}_1 & \mathrm{Z}_1 \\ \mathrm{X}_2 & \mathrm{Y}_2 & \mathrm{Z}_2 \\ \mathrm{X}_3 & \mathrm{Y}_3 & \mathrm{Z}_3 \end{vmatrix} [\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3]$$

But $[\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3] = h \cdot \triangle \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3$, and $[\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3] = h \cdot \triangle \mathrm{ABC}$ where h is the height of the point O from the plane of $\triangle$ ABC.

Hence, we have

$$\triangle\, \mathrm{P}_1\mathrm{P}_2\mathrm{P}_3 = \begin{vmatrix} \mathrm{X}_1 & \mathrm{Y}_1 & \mathrm{Z}_1 \\ \mathrm{X}_2 & \mathrm{Y}_2 & \mathrm{Z}_2 \\ \mathrm{X}_3 & \mathrm{Y}_3 & \mathrm{Z}_3 \end{vmatrix} \cdot \triangle \mathrm{ABC}.$$

4. **The Equation of a straight line.** Let $\boldsymbol{e}$ be the unit vector along one direction of a given line. Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be unit vectors along BC, CA, AB and θ_1, θ_2, θ_3 the angles between $\boldsymbol{e}$ and $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. Then if (x_1, y_1, z_1) is a given point on the line, and (x, y, z) a variable point on it, and r the distance between them, it is easily seen that

$$\frac{x - x_1}{\dfrac{a \sin \theta_1}{2\triangle}} = \frac{y - y_1}{\dfrac{b \sin \theta_2}{2\triangle}} = \frac{z - z_1}{\dfrac{c \sin \theta_3}{2\triangle}} = r,$$

where a, b, c are the sides of the triangle. These are then the equations of the line. They are written in the usual form

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r,$$

where $l + m + n = 0$, $a^2mn + b^2nl + c^2lm = -1$.

These relations are easily obtained as follows: We have

$$\overline{BC}+\overline{CA}+\overline{AB}=0 \quad \text{i.e.} \quad a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}=0.$$

Taking the vector product with $\boldsymbol{e}$, we get

$$a \sin \theta_1 + b \sin \theta_2 + c \sin \theta_3 = 0 \qquad \text{i.e.} \quad l+m+n=0.$$

Next,

$$\overline{P_1P} = r\boldsymbol{e} = \overline{OP} - \overline{OP_1} = (x-x_1)\boldsymbol{p}_1 + (y-y_1)\boldsymbol{p}_2 + (z-z_1)\boldsymbol{p}_3 = lr\boldsymbol{p}_1 + mr\boldsymbol{p}_2 + nr\boldsymbol{p}_3$$

$$\therefore \quad \boldsymbol{e} = l\boldsymbol{p}_1 + m\boldsymbol{p}_2 + n\boldsymbol{p}_3$$

$$\therefore \quad \boldsymbol{e}^2 = 1 = l(-m-n)\boldsymbol{p}_1^2 + m(-n-l)\boldsymbol{p}_2^2 + n(-l-m)\boldsymbol{p}_3^2$$
$$+ 2lm\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + 2mn\boldsymbol{p}_2\cdot\boldsymbol{p}_3 + 2nl\boldsymbol{p}_3\cdot\boldsymbol{p}_1$$
$$= -\sum mn(\boldsymbol{p}_2-\boldsymbol{p}_3)^2 = -\sum a^2mn$$

$$\therefore \quad a^2mn + b^2nl + c^2lm = -1.$$

The reader will easily verify the following:

1. If $lx+my+nz=0$ is a straight line not altogether at infinity, $al\boldsymbol{i}+bm\boldsymbol{j}+cn\boldsymbol{k}$ is parallel to the line.

2. The angle between the lines

$$lx+my+nz=0 \text{ and } l'x+m'y+n'z=0 \text{ is given by}$$

$$\tan\theta = \frac{\boldsymbol{e}\times\boldsymbol{e}'}{\boldsymbol{e}\cdot\boldsymbol{e}'} = \frac{2\Delta}{\Omega}\begin{vmatrix} l & m & n \\ l' & m' & n' \\ 1 & 1 & 1 \end{vmatrix}$$

where Δ is the area of Δ ABC, and

$$\Omega = a^2ll' + b^2mm' + c^2nn' - (mn'+m'n)bc\cos A$$
$$-(nl'+n'l)ca\cos B - (lm'+l'm)ab\cos C.$$

5. Length of perpendicular from $P_1(x_1, y_1, z_1)$ on the line $lx+my+nz=0$.

Let the foot of the perpendicular be $P(x, y, z)$ and $PP_1 = p$. Then if the given line makes angles $\theta_1, \theta_2, \theta_3$ with BC, CA, AB, the equations of PP_1 are

$$\frac{x_1-x}{\dfrac{a\sin\left(\frac{\pi}{2}+\theta_1\right)}{2\Delta}} = \frac{y_1-y}{\dfrac{a\sin\left(\frac{\pi}{2}+\theta_2\right)}{2\Delta}} = \frac{z_1-z}{\dfrac{a\sin\left(\frac{\pi}{2}+\theta_3\right)}{2\Delta}} = p.$$

Hence,

$$p = \frac{2\Delta(lx_1 + my_1 + nz_1)}{al \cos\theta_1 + bm \cos\theta_2 + cn \cos\theta_3}$$

$$= \frac{2\Delta(lx_1 + my_1 + nz_1)}{(al\,\boldsymbol{i} + bm\,\boldsymbol{j} + cn\,\boldsymbol{k})\cdot\boldsymbol{e}}$$

where $\boldsymbol{e}$ is the unit vector along the given line

$$= \frac{2\Delta(lx_1 + my_1 + nz_1)}{|al\,\boldsymbol{i} + bm\,\boldsymbol{j} + cn\,\boldsymbol{k}|}$$

$$= \frac{2\Delta(lx_1 + my_1 + nz_1)}{\Omega_\circ^{\frac{1}{2}}},$$

where $\Omega_\circ = \sum a^2l^2 - 2\sum mnbc \cos A$.

I wish to express my thanks to Mr. S. Satyanarayana Rao to whom the ideas of this note are largely due.

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On the Circle in Areal Coordinates

The object of this note is to find the conditions that the general equation

$$f(x,y,z) \equiv S \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

should represent a circle in areal coordinates and hence to deduce the radius of the circle.

1. It is enough if we find the condition that the power of a point $P \equiv (x_1, y_1, z_1)$ w.r.t. S is constant for all lines through P.

Let any line through P be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r$$

where $\qquad l + m + n = 0 \qquad (1)$

and $\qquad a^2mn + b^2nl + c^2lm = -1. \qquad (2)$

Then, the power of the point P along this line $\{l, m, n\}$ is

$$\frac{f(x_1, y_1, z_1)}{f(l, m, n)}. \qquad (3)$$

In order that (3) is a constant for all $\{l, m, n\}$

we should have $\qquad f(l, m, n) = k$ (a constant) $\qquad (4)$

for all $\{l, m, n\}$, whenever (1) and (2) hold. This means again that if l, m, n are treated as cartesian coordinates in some space, the section of (2) by (1) lies on (4). That is, the common section of (2) and (4) consists of two conics of which one of the conics lies in the plane (1).

This means that for suitable constants λ, A, B, C

$$f(l, m, n) - k + \lambda(a^2mn + b^2nl + c^2lm + 1)$$
$$\equiv (l+m+n)(Al + Bm + Cn)$$

$\therefore$ $\lambda = k$, comparing the constant terms.

Then comparing coefficients of mn, nl, and lm, we get the conditions for a circle viz.

$$\frac{B+C-2F}{a^2} = \frac{C+A-2G}{b^2} = \frac{A+B-2H}{c^2} = k.$$

2. *Lemma.* If S = 0 is a circle, then

$$\Sigma \equiv \begin{vmatrix} A & H & G & 1 \\ H & B & F & 1 \\ G & F & C & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -4D^2k^2$$ where D is the area of the triangle of reference.

For $$\frac{B+C-2F}{a^2} = \frac{C+A-2G}{b^2} = \frac{2(C+H-F-G)}{a^2+b^2-c^2} = k$$

$$\therefore \quad k^2 = \frac{4(C+H-F-G)^2 - 4(B+C-2F)(C+A-2G)}{(a^2+b^2-c^2)^2 - 4a^2b^2}$$

$$= \frac{(C+H-F-G)^2 - (B+C-2F)(C+A-2G)}{-4D^2} \qquad (5)$$

The given result may now be easily verified.

2. *To find the radius ρ of $S = 0$.*

Let (x_1, y_1, z_1) be the centre of the circle.

Then $f(x_1, y_1, z_1) = x_1\xi_1 + y_1\eta_1 + z_1\zeta_1$

$$= \xi_1 = -\frac{\Delta}{\Sigma},$$

since $\xi_1 = \eta_1 = \zeta_1 = -\dfrac{\Delta}{\Sigma}$ (Refer Askwith: *Analytical Geometry* p. 294)

The power of the centre of the circle $= -\rho^2$.

$$\therefore \quad \rho^2 = -\frac{f(x_1, y_1, z_1)}{k} = +\frac{\Delta}{k\Sigma} = -\frac{\Delta}{4D^2k^3}$$

or $$\rho^2 = -\frac{\Delta}{\frac{a^2b^2c^2}{4R^2}\{k^3\}}$$

$$\therefore \quad \rho = \frac{2R\,\Delta^{\frac{1}{2}}}{\{(2F-B-C)\,(2G-C-A)\,(2H-A-B)\}^{\frac{1}{2}}},$$

since $$ku^2 = (B+C-2F) \text{ etc.}$$

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Lightning Elimination

With the above title, a note was published by Prof. A. A. Krishnaswamy Ayyangar in *The Math. Student*, Vol. 7, p. 110 and another by T. N. Moorthy in Vol. 12, pp. 107–8. The purpose of this article is to show how the elimination in Moorthy's first problem (given below as Q. 1) can be effected in a more lightning way and also to give two other examples of elimination.

1. If $x = by + cz$, $y = cz + ax$, $z = ax + by$, prove that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

We have $$\frac{x}{by+cz} = \frac{y}{cz+ax} = \frac{z}{ax+by} = 1.$$

Now $$\frac{a}{1+a} = \frac{a \cdot 1}{1+a \cdot 1} = \frac{a\frac{x}{by+cz}}{1+a\frac{x}{by+cz}} = \frac{ax}{ax+by+cz} \qquad (1)$$

Similarly $$\frac{b}{1+b} = \frac{by}{ax+by+cz}, \qquad (2)$$

and $$\frac{c}{1+c} = \frac{cz}{ax+by+cz}, \qquad (3)$$

Adding (1), (2) and (3), the result follows.

2. If $a^x = bc$, $b^y = ca$, $c^z = ab$, show that $xyz = x + y + z + 2$.

For,

$$a^{xyz} = (bc)^{yz} = (b^y)^z(c^z)^y = (ca)^z(ab)^y = a^{y+z} \cdot b^y \cdot c^z$$
$$= a^{y+z}(ca)(ab)$$
$$= a^{y+z+2} \cdot bc$$
$$= a^{x+y+z+2}$$

Hence $xyz = x + y + z + 2$.

3. Show that the result of eliminating θ between

$$a\theta^4 + 4b\theta^3 + 4d\theta + e = 0 \text{ and } a\theta^3 + 3b\theta^2 + d = 0$$

can be expressed in the form

$$(d\sqrt{a} - ib\sqrt{e})^{2/3} - (d\sqrt{a} + i\,b\sqrt{e})^{2/3} = i\sqrt{ae}.$$

Multiplying the second equation by θ and subtracting from the first, we have, $b\theta^3 + 3d\theta + e = 0$.

Rewriting the second and the last equations in the form

$$3\theta^2 b + 1d + a\theta^3 = 0$$

$$\text{and} \quad \theta^3 b + 3\theta d + e = 0,$$

we have by the rule of cross-multiplication

$$\frac{b}{e - 3a\theta^4} = \frac{d}{a\theta^6 - 3e\theta^2} = \frac{1}{9\theta^3 - \theta^3} = \frac{1}{8\theta^3}.$$

Hence

$$d\sqrt{a} + i\,b\sqrt{e} = \left(\frac{a^{1/2}\theta^2 - ie^{1/2}}{2\theta}\right)^3$$

$$d\sqrt{a} - i\,b\sqrt{e} = \left(\frac{a^{1/2}\,\theta^2 + i\,e^{1/2}}{2\theta}\right)^3$$

$$\therefore \quad (d\sqrt{a} - i\,b\sqrt{e})^{2/3} - (d\sqrt{a} + i\,b\sqrt{e})^{2/3}$$

$$= \frac{(a^{1/2}\theta^2 + i\,e^{1/2})^2 - (a^{1/2}\,\theta^2 - i\,e^{1/2})^2}{4\theta^2}$$

$$= \frac{4i\,a^{1/2}e^{1/2}\theta^2}{4\theta^2}$$

$$= i\sqrt{ae}.$$

Remarks.

The result is equivalent to the condition that $ax^4 + 4bx^3 + 4dx + e = 0$ may have two equal roots *or* that the equations $ax^3 + bx^2 + d = 0$ and $bx^3 + 3dx + e = 0$ may have a common root. As a special case when $a = 1$, $b = p$, $d = -q$, $e = -1$ we have the following problem :—

Show that the condition that $x^4 + 4px^3 - 4qx - 1 = 0$ should have equal roots can be expressed in the form $(p+q)^{2/3} - (p-q)^{2/3} = 1$. (*Math. Tripos 1898*).

[Vide G. H. Hardy. *Pure Mathematics* 9th Ed. p. 277 Q. 24]

Trichur.

C. S. VENKATARAMAN.

The Addition Formula in Trigonometry

Two new proofs have recently been published[1] by L. J. Burton and E. A. Hedberg, for the formula

$$\sin (A+B)=\sin A \cos B+\cos A \sin B.$$

The following proof is simple and is believed to be new.

Let XOY = A, YOZ = B, so that XOZ = A + B.

Take any point Y on the common arm of the two angles, and draw a perpendicular to OY through Y to meet OX in X, and OZ in Z.

Then, $\triangle XOZ = \triangle XOY + \triangle YOZ$

$$\therefore \quad \tfrac{1}{2} OX \cdot OZ \sin(A+B) = \tfrac{1}{2} XY \cdot OY + \tfrac{1}{2} YZ \cdot OY$$

$$\therefore \quad \sin (A+B) = \frac{OY}{OZ} \cdot \frac{XY}{OX} + \frac{YZ}{OZ} \cdot \frac{OY}{OX}$$

$$= \sin A \cos B + \cos A \sin B.$$

The method is applicable for all angles A and B for which $A+B < \frac{\pi}{2}$. By writing $A = \frac{m\pi}{2} + \theta$, $B = \frac{n\pi}{2} + \phi$, where m and n are integers, and θ, ϕ are acute angles, the addition formula may be extended to all values of A and B.

The proof of the formula for sin (A − B) follows similarly, by taking the angle YOZ in a sense opposite to that of XOY.

Reference

American Math. Monthly, 56 (1949), 471—73,

Govt. College, Krishnagar. NALINI KANTA CHAKRAVARTY.

A Note on Purser's Theorem

1. A. A. Krishnaswami Ayyangar [1] has given a valuable collection of the known methods of proving Feuerbachs' theorem which says that the ninepoints circle of a triangle touches the incircle of the triangle and each of the excircles. Among these proofs is a simple one depending on a theorem of Purser which deserves to be better known than it is and runs as follows :

PURSER'S THEOREM. *If t_A, t_B, t_C are the lengths of the tangents from three points A, B, C respectively to a circle Σ, then a necessary and sufficient condition for the circle Σ to touch circle ABC is*

$$BC\cdot t_A \pm CA\cdot t_B \pm AB\cdot t_C = 0$$

where a combination of all the positive signs is obviously ruled out, but one of the other combinations holds.

This theorem is included in the following more general theorem due to John Casey.[2]

CASEY'S THEOREM. *If there are four circles denoted by 1, 2, 3, 4 and $\overline{12}$ represents the length of a common tangent to circles 1 and 2, $\overline{23}$ etc . . . having similar meanings, then a necessary and sufficient condition for the four circles to be touched by the same fifth circle is*

$$\overline{12}\cdot\overline{34} \pm \overline{23}\cdot\overline{41} \pm \overline{31}\cdot\overline{24} = 0.$$

It is clear that Casey's theorem reduces to Purser's theorem when three of the four circles 1, 2, 3, 4 reduce to point circles, and to Ptolemy's theorem for a cyclic quadrilateral when all the four circles become point circles. Purser's theorem may thus be proved, just like Casey's and Ptolemy's theorems by an inversion which reduces the fifth circle to a straight line. But in view of the fact that Purser's theorem can be used to avoid an inversion as in the proof of Feuerbach's theorem mentioned at the outset and in the proof of the theorem of § 2 below, a short and direct proof of the theorem such as the following, not depending on inversion would seem to be desirable.

PROOF OF PURSER'S THEOREM.

Adopt the notation of my enunciation of the theorem. Choose ABC as the triangle of reference in areal coordinates and write $BC=a$; $CA=b$; $AB=c$.

Then circle $ABC \equiv (a^2yz + b^2zx + c^2xy) = 0$

circle $\Sigma \equiv (a^2yz + b^2zx + c^2xy) - (t_A^2x + t_B^2y + t_C^2z)(x+y+z) = 0$

A necessary and sufficient condition for these two circles to touch is that one of them, say circle ABC should touch their radical axis $t_A^2x + t_B^2y + t_C^2z = 0$.

Now the condition for circle ABC to touch $lx + my + nz = 0$ is

$$a^4l^2 + b^4m^2 + c^4n^2 - 2b^2c^2mn - 2c^2a^2nl - 2a^2b^2lm = 0$$

or

$$a\sqrt{l} \pm b\sqrt{m} \pm c\sqrt{n} = 0.$$

Hence we get the required condition from the above by putting $l = t_A^2$; $m = t_B^2$; $n = t_C^2$.

2. The following interesting theorem has been proved by E. P. Lewis [3] by the method of inversion.

THEOREM. *If the altitudes of any triangle are drawn through the incentre and excentres, then the twelve circles on these altitudes as diameters all touch the circumcircle.*

Purser's theorem can be used to prove the above in the following manner.

Let ABC be the triangle and let $c > b$. Let P_1Q_1 be one of the "altitudes" of the triangle through the incentre, say, the line through the incentre equal and parallel to the altitude through A, P_1 being its intersection with BC, and Q_1 its intersection with the parallel to BC through A.

Then if we take the circle Σ of Purser's theorem to be the circle on P_1Q_1 as diameter, we have (writing $a+b+c=2s$)

$$t_A = AQ_1 = c \cos B - (s-b),$$
$$t_B = BP_1 = (s-b), \quad t_C = CP_1 = (s-c).$$

Consequently

$$a\,t_A - b\,t_B + c\,t_C = ac \cos B - a(s-b) - b(s-b) + c(s-c)$$
$$= \frac{1}{2}(a^2+c^2-b^2) - \frac{a}{2}(c+a-b) - \frac{b}{2}(c+a-b) + \frac{c}{2}(a+b-c) = 0.$$

Purser's theorem now shows that the circle on P_1Q_1 as diameter touches the circumcircle of triangle ABC. Similarly the circles on the other "altitudes" of our theorem as diameters touch the circle ABC.

Applying the theorem of § 2 to the triangle formed by the middle points of the sides of the triangle ABC, and using Feuerbach's theorem, we get at once the

COROLLARY. *The Nine points circle of a triangle touches each of sixteen circles related to the triangle, four of the sixteen circles being the incircle and the three excircles.*

References

(1) A. A. Krishnaswami Ayyangar: "The theorem of Feuerbach" *Math. Student*, Vol. 9 (1941) pp. 16–29.

(2) John Casey: *A Sequel to the first six books of the elements of Euclid* (London 1886) pp. 103–4.

(3) E. P. Lewis: 'On a set of twelve circles all touching the circumcircle.' *Mathematical Gazette* Vol. 14 (1929) p. 421.

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S. SITHARAMAN,
Student, V Hons.

Astatic Equilibrium

The following is an interesting way of obtaining conditions for coplanar forces in astatic equilibrium. To start with, we note that any vector is turned through an angle θ, when multiplied by Cis θ.

Let F_1, F_2, F_3, . . . , F_n be the forces acting at the points z_1, z_2, z_3, . . . , z_n.

Then, if the forces are in equilibrium, $\sum F_i$ is equal to zero, and $R(i\sum zF')=0$; where F' is the conjugate of F.

Let each force be turned through an angle θ in the same sense.

Then the forces are F_1 Cis θ, . . . , F_n Cis θ acting at the points z_1, . . . z_n. If the forces are still in equilibrium,

$\sum F$ Cis $\theta = 0$ or Cis $\theta \sum F = 0$, which is true whatever θ be.

$$\text{Also, } R(i\sum zF' \text{ Cis}'\,\theta)=0.$$

$$\begin{aligned}\text{Now, } R(i\sum zF' \text{ Cis}'\,\theta)&\\ &= R(i \text{ Cis}'\,\theta \sum zF')\\ &= R(i\sum zF') \text{ Cos }\theta + R(\sum zF') \text{ Sin }\theta\\ &= R(i\sum zF') \text{ Cos }\theta + I(i\sum zF') \text{ Sin }\theta.\end{aligned}$$

We have $R(i\sum zF')=0$.

Therefore, if in addition, $I(i\sum zF')=0$, there is astatic equilibrium. $I(i\sum zF')$ is called the VIRIAL of the system.

Therefore, the necessary conditions for astatic equilibrium are $\sum F=0$ and $\sum zF'=0$.

We can show that these are sufficient.

Hindu College, Guntur.

N. V. SUBRAHMANYAM.

A Note on the Cumulants of a Binomial Distribution

J. B. S. Haldane,[1] has obtained a simple recursion formula for the cumulants of a Binomial distribution. The purpose of the present note is to give an alternative proof of the recursion formula which seems to be simpler.

The characteristic function of the Binomial distribution $(p+q)^n$, which generates moments about its mean is given by

$$\phi(t) = (qe^{ipt}+pe^{-iqt})^n.$$

The $(r+1)^{\text{th}}$ cumulant k_{r+1} is given by the formula

$$k_{r+1} = \frac{1}{i^{r+1}} D^{r+1} [\log \phi(t)]_{t=0} \text{ where } D \equiv \frac{d}{dt}$$

$$= \frac{1}{i^{r+1}} D^r \left[\frac{nipq\,(e^{ipt} - e^{-iqt})}{qe^{ipt} + pe^{-iqt}}\right]_{t=0}$$

$$= pq \cdot \frac{1}{i^r} D^r \left[\frac{n(e^{ipt} - e^{-iqt})}{qe^{ipt} + pe^{-iqt}}\right]_{t=0} .$$

We have

$$k_r = \frac{1}{i^r} D^r \left[n \log (qe^{ipt} + pe^{-iqt})\right]_{t=0}$$

$$\therefore \quad \frac{dk_r}{dq} = \frac{1}{i^r} D^r \left[\frac{d}{dq} \{n \log (qe^{ipt} + pe^{-iqt})\}\right]_{t=0}$$

$$= \frac{1}{i^r} D^r \left[\frac{n\,(e^{ipt} - e^{-iqt})}{qe^{ipt} + pe^{-iqt}}\right]_{t=0}$$

$$\therefore \quad k_{r+1} = pq \frac{dk_r}{dq} .$$

Reference

[1] J. B. S. Haldane: *Biometrica* Vol. 31 (1939) pp. 392–393.

University of Bombay. V. N. MURTI.

QUESTIONS AND SOLUTIONS

It has been decided to revive the "Questions and Solutions" Section, and we trust this section will stimulate the keen interest of a large number of teachers and students whom the *Mathematics Student* primarily intends to serve. Contributors to this section will kindly note the following:

(*1*) *Questions should be new. New solutions and discussions of known results will come under a separate section.*

(*2*) *The Questions should be of common interest to a wide class of teachers and students, and should ordinarily be within the standard of the B.Sc. Honours and M.Sc. Syllabuses of Indian Universities.*

(*3*) *Questions on highly specialized topics should be avoided.*

(*4*) *Authors are requested not to send in the form of a 'Question' any preliminary result or lemma that may occur in the course of some investigation. Such a result will naturally be published with that investigation, and need not be published again separately.*

(*5*) *No Question for solution will be entertained, unless accompanied by the author's complete solution, in a form ready for the press.*

(*6*) *All Solutions should be written out legibly, using one side of the paper only, in a form suitable for publication.*

THE EDITOR.

QUESTIONS FOR SOLUTION

1818 (S. Sitharaman). ABC is a plane triangle. With the usual notation, show that

$$\sin^{-1}\left[\cos\frac{A}{2}\sqrt{1-\frac{a}{s}}\right]+\sin^{-1}\left[\cos\frac{B}{2}\sqrt{1-\frac{b}{s}}\right]+\sin^{-1}\left[\cos\frac{C}{2}\sqrt{1-\frac{c}{s}}\right]=\frac{\pi}{2}.$$

1819 (S. Sitharaman). The harmonic polar of the symmedian point w.r.t. a Δ ABC meets BC, CA, AB in D, E, F. If AD, BE, CF are the lines $L=0$, $M=0$, $N=0$, the coordinates being areal, show that the conic

$$\frac{\cot\frac{A}{2}}{L}+\frac{\cot\frac{B}{2}}{M}+\frac{\cot\frac{C}{2}}{N}=0$$

is a Fragnon ellipse if $\cos A+\cos B+\cos C=\sqrt{6}-1$, and that one focus is at the circumcentre of Δ ABC.

[In a Fragnon ellipse, the minor auxiliary circle passes through the real foci ; $e=1/\sqrt{2}$]

1820 (S. Sitharaman). If $f(p, q, r) = (m-n)(m+n-l)p + (n-l)(n+l-m)q + (l-m)(l+m-n)r$, verify that

$$f(\xi p+\eta q+\zeta r,\ \eta p+\zeta q+\xi r,\ \zeta p+\xi q+\eta r)=(\xi+\eta+\zeta)f(p, q, r).$$

Interpret geometrically.

ANNOUNCEMENTS AND NEWS

We deeply regret to report that Dr. S. Sivasankaranäräyana Pillai (S. S. Pillai), D.Sc. lost his life in an airplane accident near Cairo on the 31st August 1950. At the time of his death, he was on his way to the United States of America to attend the International Union Conference of Mathematicians as one of the three delegates from India. He was to have spent the Academic year' 1950—'51 in the Institute for Advanced Study, N.J.

A detailed obituary Notice will appear in a subsequent issue of the *Mathematics Student.*

Dr. K. Chandrasekharan, of the Tata Institute of Fundamental Research, Bombay was deputed by the Government of India to participate in the International Congress of Mathematicians held at Cambridge, Mass., U.S.A. from the 30th August to the 6th September 1950.

Dr. Ram Behari, Dr. A. N. Singh, and Mr. K. Balagangadharan represented the Indian Mathematical Society at the Symposium on "History of Science in South Asia," organized by the National Institute of Sciences, India at Delhi from the 5th to the 7th of November 1950.

Dr. S. M. Shah of Aligarh has proceeded to Oxford for study and research in Mathematics, under a Government of India Research Scholarship.

We thank Sri. C. Bhaskarayya (recently retired from the Indian Audit and Accounts Service) for his gift of some back volumes of the *Journal of the Indian Mathematical Society*, and of the *Mathematics Student*, to the Library of the Indian Mathematical Society.

The Ramanujan Institute of Mathematics was inaugurated last April by Dr. Sir A. Lakshmanaswami Mudaliar, Vice-Chancellor of the University of Madras. This is founded by Dr. Sir Rm. Alagappa Chettiar, M.A., D.Litt. and Dr. T. Vijayaraghavan, D. Phil. (Oxford), is the Director. This is the first institute in India to devote itself purely to research in Mathematics and is at present located in 'Krishna Vilas', Vepery, Madras 7.

The Library of the Society is now functioning at "Krishna Vilas", Vepery, Madras 7. Dr. T. Vijayaraghavan is in charge as the Librarian.

The following have been admitted as Life Members of the Society :—

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,, S. Krishnan, M.A., Asst. Professor, Presidency College, Madras.

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,, T. Natarajan, M.A., Professor of Maths., American College, Madura.

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,, M. N. Khatri, Headmaster, Kandari School, Kandari, Baroda.

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Errata for Volume XVII—*Mathematics Student*

	For	*Read*
P. 35. 6th line from bottom :	$p^{m-\delta} - p^{m-\delta} - p^{m-\delta-1}$	$p^{m-\delta} - p^{m-\delta-1}$
P. 44 (opposite group photo) :		
In first line	M. Srirangamma	K. N. Kamalamma
In sixth line	N. Kamalamma	M. Srirangamma

REVIEWS

Lattice Theory

By GARRETT BIRKHOFF. *Revised Edition* (1948).
American Mathematical Society's Colloquium Publications. Vol. XXV pp. 283.

The appearance of this revised edition of G. Birkhoff's *Lattice Theory* bears evidence to the increasing recognition of the usefulness of the lattice approach in various mathematical fields. This book reveals how such a study is illuminating not only in Algebra proper but also in set topology, in the study of function spaces, in the axiomatics of Geometry, in logic and in probability. The extensive references to the literature will be particularly welcome to the mathematical worker. There is also a large collection of problems distributed through the chapters.

The necessary preliminary definitions and examples are covered by the forewords on Algebra and Topology, and the first chapter 'Partially Ordered Sets.' Then follows a study of the various types of lattices giving their characteristics, their algebraic features considered as abstract algebras, and typical examples of such lattices. Thus there are chapters on Lattices (ch. II), Complete Lattices (ch. IV), Modular Lattices (ch. V), Semi-Modular Lattices (ch. VII), Complemented Modular Lattices (ch. VIII), Distributive Lattices (ch. IX), and Boolean Algebras (ch.X). A chapter on chains and chain conditions (ch. III) deals with the well-ordered set, various forms of the axiom of choice and the intrinsic topology of the chain. Topological structures for lattices are introduced by convergence of directed sets (ch. IV) and by valuations (ch. V). Finally, it is in this edition that appear the Lattice-ordered Semi Group and Lattice-ordered Group (chs. XIII and XIV)—two algebraic structures possessing both an order structure and a group multiplication.

Among 'Applications to Algebra' (ch. VI) occur generalizations of the Jordan-Holder theorem to algebras with permutable congruence relations, and one element subalgebras, and subdirect union resolutions. Again the lattice-ordered semigroup provides the way to the study of the order and multiplication structures of families of ideals of rings.

An abstract form of measure theory is introduced in the chapter on Boolean algebras and is continued in the chapter on applications to set theory (ch. XI); a chapter on Vector Lattices (ch. XV) considers the order structure of function spaces.

The chapter on complemented modular lattices gives the lattice approach to the axiomatics of projective geometries, and a chapter deals with applications to logic and probability (ch. XIII). Probability theory is continued in the last chapter (ch. XVI) on Ergodic theory wherein Markoff's processes are studied.

V. S. K.

An introduction to the Theory of Mechanics

By K. E. BULLEN, M.A., SC.D., F.R.S., *Prof. of Applied Mathematics, University of Sydney*

Sydney Press, Sydney, 1949, pp xvi + 368

This exceedingly well written book covers roughly the courses in Dynamics, Statics and Hydrostatics at the pass course level. It contains 18 chapters of which 7 are devoted to particle dynamics (motion in a straight line and in a circle, parabolic motion under gravity, impact, machines, and simple, two-dimensional motion), 3 chapters to Rigid Dynamics (moments of inertia, simple two-dimensional motion), 1 to two-dimensional Statics and 1 to Hydrostatics including metacentre and gases). Calculus methods are used. The vector notation is explained and used to point out the formal analogy to one-dimensional motion when vectors are used. An appendix extends the use of vectors to obtain the mass acceleration of a point referred to rotating frames and to derive the additional forces [Centrifugal, Cariolis and $MR \times \partial\Omega/\partial t$] and to demonstrate Newtonian principle of relativity. The presentation is clear and logical and strikes a good balance between pure mathematics and emperism in laying the foundations of the subject. "Remarks" are inserted at frequent intervals drawing the attention of the student to crucial points which he is likely to miss.

There are several novel features in the book which go to enhance its value. The very first chapter contains "some remarks about applied mathematics" which bring out important features of scientific method and the roles of induction and deduction in Pure and in Applied mathematics and the limitations on the correspondence sought to be set up between natural phenomena and the mathematical structure erected for their representation. There are 3 "interludes" where, at different stages the author reviews the concepts and methods introduced so far so that "the forest is not obscured by the trees." On the other hand, there is a "glimpse forward" at the end of the book where there is a brief survey of the whole range of Applied Mathematics dealing with further elementary mechanics, Hydrodynamics, Elasticity, Hamiltonian Mechanics, Electromagnetism, waves, Thermodynamics, Statistical mechanics, Relativity and Quantum mechanics. While one may wonder what kind of reaction this comprehensive survey, compressed into 11 pages will evoke from the learner studying elementary dynamics for the first time, such a "glimpse forward" should be warmly welcomed for two reasons. Firstly no text-book should in the reviewer's opinion, be written as if the subject had no loose ends and the last word said at the last page; secondly, a text-book should have something of interest and value to the teacher as well as for the learners. From this latter point of view, I am very glad to find the author adding to the taste of the treatment by S.P.I.C.E's, ("special points in choosing examples") which follow almost every set of worked examples, and explain the points for illustrating which the examples have been chosen. Though apparently addressed to the student, these should be specially helpful to the teacher. There are a large

number of exercises, several of them coined by the author, the more difficult of which are marked by an asterisk. The book abounds in homely advice to the student, including which formulae to commit to memory, and even advice in answering examination papers!

Altogether a valuable addition to the text-books on the subject at this level.

A. NARASINGA RAO.

Mechanics via the Calculus

By P. W. NORRIS AND W. S. LEGGE. *Third Edition, Revised and enlarged.* Cleaver-Hume Press, London, 1950. pp. 367, 16 sh. net.

This book which is well-known and widely used as a text-book and book of reference in technical colleges has now come out in its third edition. The features of this edition are the addition of a chapter with the title "An Introduction to Vector Analysis," two appendices on the principal axes of a body and the instantaneous centre of rotation, and the addition of 58 miscellaneous problems for solution. The additional matter is welcome, but one wonders why in these days when the teaching of applied mathematics has undergone radical reforms, vector analysis should be given as an introductory chapter *at the end* of the book, instead of introducing vector methods right from the beginning of the book.

The book, embodying the principal contents of the subject-matter of dynamics, statics and hydrostatics, and with its characteristic collection of worked examples, will continue to be a valuable and popular book with the teachers and the students of the Engineering Colleges, as also of the Mathematics Honours classes.

C. N. S.

A Text Book of Statistics

By V. SEETHARAMAN, M.A., M.Sc., Ph.D. *Prof. of Mathematics Thyagaraja College, Madura.*
Mangalodayam Press, Trichur. pp. 268. Rs. 5/-.

The author says in the preface that the book is designed to meet the long felt need of the B.A. and B.Sc. students of the South Indian Universities for a text book, that will cover exactly, the syllabus in statistics. Evidently the author has in mind the three Universities of Madras, Travancore and Annamalai. The book is mainly descriptive and elementary. The first few chapters deal with 'definition of statistics', 'organisation of statistical investigation', 'graphical representation', 'Indian Census', 'Averages', 'Dispersion' and 'Moments'. There is no mention even of β-coefficients and measure of kurtosis. Only the method of least squares is used in curve fitting and there is no mention of test for goodness of fit.

The next chapter is on correlation and regression. This includes the product-moment r, rank correlation and correlation ratios. Four small chapters are devoted to 'probability' and the three distributions—the Binomial, the Normal and the Poisson. Only simple properties of these distributions are considered. Recurrence formula for moments, normalising means and medians are omitted.

There are chapters on Sampling, Index numbers and Mortality Table. Formulae of various tests are given without mathematical proofs and examples are worked out. Both large and small samples are treated. Since tests for small samples are valid for large samples and not vice versa, a number of tests might well have been omitted. One wonders at the omission of the variance ratio test.

The Appendix includes University question papers, the Normal Probability integral Table, and life Table of Males (All India-Census, 1931). At the end of each chapter, simple exercises are given for practice.

The book suffers from the omission of a number of important topics which any student of modern statistics ought to know. While it may not serve as a comprehensive introduction to the study of the subject, it is hoped that it will be a useful text book for the students for whom it is intended.

K. V. K.

First year Calculus

By M. N. BHAT

Aryabhusan Press, Poona. pp. 190. Rs. 3/-

This is intended to serve as a text book for the First year Course of the Poona University, and will approximately cover the Intermediate syllabuses of some of the other Universities. The notions of limit and differential coefficient are very clearly explained with examples, and are followed by chapters on differentiation, and its geometrical and physical applications. Integration is, dealt with in two chapters.

The book could have been made a little more complete by including some more topics, e.g. the derivative when x and y are expressed in terms of t, differential coefficients of higher orders, definite integrals etc.

The following is a defect which stands out in an otherwise well-written book. In working out the differential coefficients of the inverse sine and cosine functions, no explanation is given as to how the ambiguity in the sign is to be removed. The ambiguity can be removed only by considering the principal values of the functions.

C. N. S.

1. *Co-ordinate Geometry*
(1949) Pages 1–180 Part I & 1–294 Part II.

2. *Solid Geometry*
(1949) pp. 1–212 Rs. 3–12–0

3. *Hydrostatics*
(1950) (pp. 1–298)

By D. K. SEN, M.SC., Ph.D., *Rajaram College, Kolhapur.*
(Educational Publishing House, Bombay)

1. The Text-book of Co-ordinate Geometry is intended for the use of students of the Intermediate class. The subject is dealt with in the familiar and well trodden path and is in two parts. The First part deals, in five chapters of 150 pages, with the earlier portions of the subject and takes the student to the end of the chapter on Pairs of Straight Lines. Part II (Chapters VI to IX; pp. 1–294) deals with the Circle, the Parabola, and The Ellipse and the Hyperbola referred to their principal axes.

There are elaborate explanations of the steps in the work (savouring of the black-board work in the class room). Alternative proofs are provided and a large number of worked examples are given. A much larger number of examples (Answers & hints covering 40 pages 254–294) are intended as exercise to the student who is introduced to the subject for the first time.

In a subject like this where there are a number of available text-books there are not many special points that can be indicated. A few like the following may, however, be observed. On page 9 part II, the property of the angle in the semi-circle can be used to determine the equation of the circle on the line joining two points as diameter instead of the verification method as given. Again on page 137, the equation to the ellipse with the major axis along the y-axis proved as therein makes the subject appear more voluminous than it need be.

There are not many noticeable printing errors.

Considering the class for which this book is intended, there is room for the feeling that the young student who is introduced to the subject through this book, is likely to find this exhaustive volume a bit exhausting. It is hoped that, with the necessary help of a teacher to guide him through the essentials of the subject, this book will be useful as a text-book for the students for whom it is intended.

2. Solid Geometry.

In the 200 pages of the book, the student is taken steadily through the subject and there are not many noticeable points of departure from the familiar path. On page 37, property 15, it may be felt that the proof by using Pythagoras's Theorem might have been given with advantage; at any rate as an alternative method. The book will be found useful for the Intermediate class.

3, Hydrostatics.

In the Eleven chapters of the book, the subject-matter is dealt with in a natural order. A number of illustrative examples add to the usefulness of the book. The printing and get-up of the book are very attractive. The book is sure to be of great help to the students for whom it is intended.

M. J. N.

Kona Ganitham (Trigonometry)

By T. K. MANICKAVACHAGAM, M.A.,L.T., *Professor of Mathematics, Dr. Alagappa Chettiar's College, Karaikudi.*

This is one of a series of books published by the Annamalai University to make it possible for Tamil to be used as the medium of instruction in the collegiate classes, and was written by the author when he was lecturer in mathematics at Annamalainagar. It covers the course in Trigonometry at the intermediate level up to and including sub-multiple angles, solution of trigonometrical equations, maxima and minima of Trigonometric functions etc. The definitions and explanations are all in Tamil, but the author has done wisely in retaining English lettering for the figures, and the usual symbols in English for sine, cosine etc. so that a page of mathematical working looks just like a page taken out of an English text book.

The author introduces the sine, cosine etc. of the general angle by the definitions $\sin \theta = y/r$, $\cos \theta = x/r$ etc. where x, y, r relate to a point on the radius vector. Since the student is already familiar with the signs of coordinates of a point in the various quadrants, this mode of presentation is easy to grasp and avoids the use of projections. There are worked examples and exercises. A glossary of English mathematical terms with Tamil equivalents is given at the end of the book. A complete list of formulæ obtained in the book is given in the beginning with page references, but without an indication as to which of them are worth memorizing.

The book should meet admirably the purpose for which it was written.

A. NARASINGA RAO.

BOOKS RECEIVED FOR REVIEW

W. V. D. HODGE & D. PEDOE: *Methods of Algebraic geometry*, Vol. I. Cambridge University Press, 1947 pp. 440.

ANDRE WEIL: *Foundations of Algebraic Geometry.* American Mathematical Society Colloquium publications Vol. XXIX, 1946, pp. 288.

RAYMOND LOUIS WILDER: *Topology of manifolds,* Amer. Math. Society Colloquium Publications, Vol. XXXII, 1949, pp. 402.

JACOB STEINER: *Geometrical constructions with a ruler, given a fixed circle with its centre.* Translated from the original German by Marion Elizabeth Stark, with introduction and Notes by R. C. Archibald, Scripta Mathematica, Studies No. 4, New York, 1950, pp. 88.

L. SCHWARTZ: Theorie des distributions Tome I. Actualites Scientifiques et Industrielles, 1091, Hermann and Cie, Paris, 1950, pp. 148.

J. L. WALSH: The Location of Critical Points of Analytic and Harmonic Functions. Amer. Math. Soc. Colloquium Publications, Vol. XXXIV, $ 6-00.

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Publications of the Indian Mathematical Society

1. Memoir on Cubic Transformations Associated with a Desmic System, by Dr. R. Vaidyanathaswami, pp. 92—price Rs. 3.

2. Table of Partitions, by Dr. Hans Raj Gupta, pp. 81, price Rs. 5.

3. The Journal of the Indian Mathematical Society.

4. The Mathematics Student.

The *Student* and the *Journal* appear quarterly. The annual subscription is Rs. 6 for each. A single issue costs Rs. 1-8-0.

Applications for membership and all business communications should be addressed to Prof. S. Mahadeva Iyer, M.A., L.T., 2, Bhaskarapuram, Mylapore, Madras.

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